A.P. Calculus Chapter 6 Mechanical 2002

			EXAM CHAPTER 5 # 1 R/C Pg 166 - 171; comment and/or explain EACH of the 7 examples. **Be *very* thorough!** **Topic: Rolle's thm** 10/31	# 2 Pg 171 - 172 1, 3, 5, 8 - 16, 19 #10 "D" ans: $Sec^{-1}\ 2x$ # 19 is important Explain # 5 carefully **Lecture: Rolle's Thm** 11 / 1
# 3 Pg 172 20 Pg 177 1, 3, 6 - 9, 11 - 14, 16, 17 Pblms 16 and 17 are very important! **Law of the Mean** 11 / 4	# 4 Pg 177 - 178 18 - 21 Hint 18: see pg 83 Hint 20A multiply by (1+x) Pblm 20 C use calculator to get all 3 digits **Lecture: Pblms 20, 16, 17 from HW # 3** 11 / 5	# 5 Pg 200 1, 4, 5, 8, 9, 25 - 32 For pblms 25 - 32; ALSO state coordinates for *ALL* extrema. **Max, Min, PI, graphing** **#19 pg 182** 11 / 6	182 #19 # 6 Pg 178 22, 23 # 7 Pg 181- 182 1 - 14, 16 - 22 **Limits. prove** $\sqrt{ab} < \frac{1}{2}(ab); 0 < a < b$ **on blue paper** 11/ 7	R/C Pg 232 - 239 # 8 • • • • • • • • • • • • • • • Pg 239 1 - 20 # 9 **l'Hopital's Rule finally** 11/ 8
No School today # 10 Pg 215 2, 3, 4, 7 - 9, 15, 18 for pblm 18 only, make f, f', f'' graphs **More graphing** **Uses second derivative**	# 11 Pg 239-240 21 - 24, 26 - 35, 39 ***ALSO*** Find: the two limits listed at the bottom of the sked **Put 2,3,4, 18 on boards explain everything. number lines, limits, etc.** **also pg 200 1-5, 25-32**	# 12, 13 GQ Ch6 Pg 241 - 242 1 - 3, 5, 7 GQ Ch6 Pg 241 - 242 10 - 13, 19AB GQ Ch 1 - 6 Comp Pg 243 1, 2, 3 **have HW11 on boards** 11/ 13	# 14 GQ Ch 1-6 Pg 243 4 - 9, 11 - 15 GQ Ch 1-6 Concepts Pg 244 3, 4, 5 # 15 RE Ch 1 - 6 Pg 244 - 249 1 - 26, 33, 34, 44 11/ 14	**# 16, 17, 18, 19,20** **Pick up** **Continuous Enjoyment:** ***Four Corners of the World* edition** 11/ 15 **On the day when the multiple choice problems are handed out.**
Put up pblms 1 - 20 11/18	Put up 21 - 40 11/ 19	Put up 41 - 64 11/ 20	Put last year's exam on the chalk boards. 11/ 21	**Show detail of pblm 13 from pg 242. Most students didn't use # lines properly or at all** **Exam 11/22** 11/22
	$\lim_{x \uparrow \pi} (1 - \sin x)^{\cot x}$	$\lim_{x \uparrow \frac{\pi}{2}} (1 + \cos x)^{\tan x}$	**Find those analysis notes !!!!!!** **I *just remembered* a few things we didn't cover last year.** **You're going to *just* love this!**	

(1) a) Rolle's Thm states that $\exists\ \overline{x} \in (0, 2\pi) \Rightarrow f'(\overline{x}) = 0$

b) $f'(x) = -\sin x \overset{\text{Set}}{=} 0 \quad \overline{x} = \pi$

(2) a) $\exists\ \overline{x} \in [-1, 1] \Rightarrow f'(\overline{x}) = 0$

b) $f'(x) = 3x^2 - 1 \overset{\text{Set}}{=} 0$

$x^2 = \frac{1}{3}$

$x = \pm \frac{\sqrt{3}}{3}$

(3) $f'(x) = 4x^3 - 4x \overset{\text{Set}}{=} 0$

$4x(x^2 - 1) = 0 \qquad x \in \{-1, 0, 1\}$

(4) a) 1 b) $\frac{(2k+1)\pi}{2}$ $k = 1, 2$ c) $f'(x) = \cos x \Big\}_{\frac{(2k+1)\pi}{2}}$ does $= 0$.

(5) a) $\frac{\sqrt{2}}{2}$ b) $\frac{\pi}{4}$ c) NO D) NO, max is at an endpt,

MAX Extrema occur at end pts as well as at critical values of x

(6) a) $f(x) = 5 + (x - x^2)e^x$

$f(0) = 5 \qquad f(1) = 5 + (1-1)e^1 = 5$

$f'(x) = (1-2x)e^x + (x - x^2)e^x \overset{\text{Set}}{=} 0$

$x^2 + x - 1 = 0 \qquad x = \frac{-1 \pm \sqrt{1+4}}{2}$

$x = \frac{-1 + \sqrt{5}}{2} \qquad x = \frac{-1 \pm \sqrt{5}}{2}$

(8) $f(x) = x^3 - 3x$

a) $f'(x) = 3x^2 - 3 \overset{\text{Set}}{=} 0 \quad x = \pm 1$

b) Since $x = 1$ is an end pt, we only know that the max occurs at either 1, or 5. We must check each.

c) $f(5) = 110$,

(9) $f(x) = \sqrt[3]{x^2} = (|x|)^{\frac{1}{3}}$ derivative doesn't exist throughout the $(-1, 1)$. Since $f'(x) = \frac{2}{3}x^{-\frac{1}{3}}$ not defined at $x = 0$

171

(10) a) $x \tan^{-1} 5x - \frac{1}{10} \ln(1+25x^2)$

$$\tan^{-1} 5x + \frac{5x}{1+25x^2} - \frac{1}{10} \cdot \frac{50x}{1+25x^2} = \tan^{-1} 5x$$

b) xe^{-x^2} $\quad xe^{-x^2} + (-2x)xe^{-x^2}$

$$xe^{-x^2}(1-2x^2)$$

c) $\sqrt[3]{5+2x}$ $\quad \frac{2}{3(5+2x)^{2/3}}$

$$\frac{8x}{2\sqrt{4x^2-1}}$$

D) $x \sec^{-1} 2x - \frac{1}{2} \ln(2x + \sqrt{4x^2-1})$

$$\sec^{-1} 2x + \frac{2x}{|2x|\sqrt{4x^2-1}} - \frac{1}{2(2x+\sqrt{4x^2-1})}\left(2 + \frac{4x}{\sqrt{4x^2-1}}\right)$$

$$\sec^{-1} 2x + \frac{2x}{|2x|\sqrt{4x^2-1}} - \frac{(2\sqrt{4x^2-1} + 4x)}{(4x+2\sqrt{4x^2-1})\sqrt{4x^2-1}}$$

$$\sec^{-1} 2x + \frac{1}{\sqrt{4x^2-1}} - \frac{1}{\sqrt{4x^2-1}} =$$

$\sec^{-1} 2x$

e) $\frac{\sin 3x}{\sqrt{1+4x}}$ $\quad \frac{-3\sqrt{1+4x}\cos 3x - \frac{2 \cdot \sin 3x}{(1+4x)^{3/2}}}{1+4x}$

$$\frac{-3\sqrt{1+4x}\cos 3x}{1+4x} - \frac{2\sin 3x}{(1+4x)^{5/2}}$$

$$\frac{-3\cos 3x}{\sqrt{1+4x}} -$$

f) $\sqrt{1+3\cos^2 5x}$

$$\frac{30\cos 5x}{2\sqrt{1+3\cos^2 5x}} = \frac{-15\cos 5x \sin 5x}{\sqrt{1+3\cos^2 5x}}$$

(11) a) nothing because f is not continuous $[a,b]$
b) see below

(12) $f(x) = e^{-x}\sin x$

$f'(x) = -e^{-x}\sin x + e^{-x}\cos x \overset{\text{set}}{=} 0$

$\sin x = \cos x \qquad [0, 2\pi)$

$\frac{\pi}{4}, \quad \frac{5\pi}{4}.$

(13) Proof Let f be defined on $[a,b]$. If f has a min value at $\overline{X} \in (a,b)$ & if $f'(\overline{X})$ exists, then $f'(\overline{X}) = 0$,

$f(\overline{X})$ is min, then $f(\overline{X}+h) > f(\overline{X})$

$h > 0$

$$\frac{f(\overline{X}+h) - f(\overline{X})}{h} \leq 0$$

if $h < 0$

$$\frac{f(\overline{X}+h) - f(\overline{X})}{h} \geq 0 \qquad \text{so}$$

$$0 \leq \frac{f(\overline{X}+h) - f(\overline{X})}{h} \leq 0 \quad \therefore \quad \frac{f(\overline{X}+h) - f(\overline{X})}{h} = 0$$

$\therefore f'(\overline{X}) = 0$

(14) a) No curve continuously has maximum as it gets closer to $x=2$ so there is no max.

b) Yes at 0. No Same reason as a.

#11 B

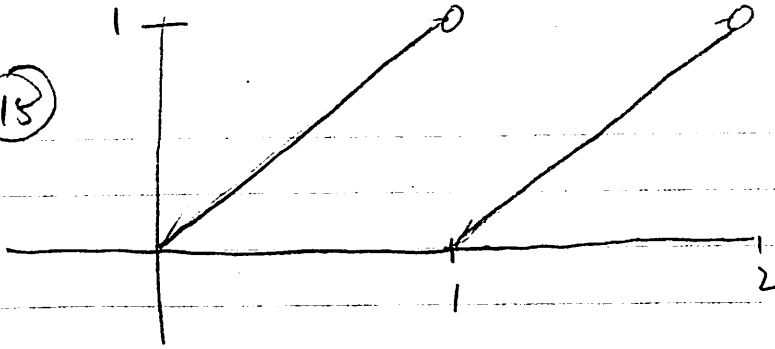
$f(x) = \frac{x^2-4}{x+\frac{1}{2}} \qquad f'(x) = \frac{(x+\frac{1}{2})2x - (x^2-4)}{(x+\frac{1}{2})^2} = \frac{2x^2+x-x^2+4}{(x+\frac{1}{2})^2} \overset{\text{set}}{=} 0$

$x^2+x+4 \overset{\text{set}}{=} 0 \qquad x = \frac{-1 \pm \sqrt{1-16}}{2}$ reject

$\therefore$ NO values.

Not Necessary
f Not Continuous (at $x = -\frac{1}{2}$) throughout interval $\therefore$ No $\overline{X}$ values

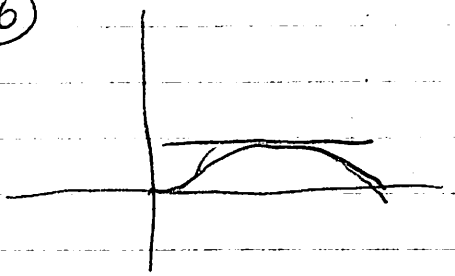
(15)



b) No max because f approaches 1 as $x \to \infty$.

c) No fctn is not continuous.

(16)



Two roots of polynomial imply $f(a) = f(b) = 0$ So there is a (min) max, at max or min $f'(x) = 0$ if Root of Derivative

Note: Since f is polynomial, we know derivative exists $\forall\ x \in D_f$

(19) Christelle Thomas Pg 130 Combine Rolle's thm w/ Intermediate Value thm.

important

a, b are Real #'s $\Rightarrow$

a) $f(x)$ is continuous on $[a,b]$ & its 1st derivative $f'(x)$ exists on (a,b)

b) $f(a)$ & $f(b)$ have opposite signs

c) $f'(x) \neq 0$ for any $x \in (a,b)$

Then $\exists$ one & only one Real Root if $f(x) = 0$, between a, b

(19) $x^5 + 2x^3 - 2 = 0$ 1st Show at least 1 then show at most 1 then conclude Exactly 1 Sol.

Define $f(x) = x^5 + 2x^3 - 2$

1 Zero of the fctn

$f(0) = -2$, $f(1) = 1$ ∴ At least 1 Sol by Intermediate Value Thm in $[0, 1]$.

Next suppose there are 2 distinct Solutions.
Let the two solutions be $p \neq q$. So
$f(p) = 0 = f(q)$
By Rolle's thm there is at least one $\overline{x} \in (a, b)$ such that $f'(\overline{x}) = 0$.

$f'(x) = 5x^4 + 6x^2 = x^2(5x^2 + 6)$

Notice $f'(x) > 0 \quad \forall x \in (0, 1)$ ∴ there is no $\overline{x} \in (0, 1)$ ⇒ $f'(\overline{x}) = 0$ ∴ p, q cannot exist so there is at most one solution as promised by the Intermediate Value thm.

Since there is at least 1 and at most 1 solution, there is exactly 1 solution.

thm Let a, b be real #s AND:
a) $f(x)$ is continuous on $[a, b]$ & its first derivative, $f'(x)$ exists over (a, b).
b) $f(a)$ and $f(b)$ have opposite signs
c) $f'(x) \neq 0$ for ANY $x \in (a, b)$ (i.e. f doesn't "turn around.")

Then ∃ one & only one Real Root, if $f(x) = 0$ between a and b.

Pg 172

20 $3\tan x + x^3 = 2$ $\quad \left[0, \frac{\pi}{4}\right]$

define $h(x) = 3\tan x + x^3 - 2$

$h'(x) = 3\sec^2 x + 3x^2$

Obviously $h'(x) > 0$ for $x \in (0, \frac{\pi}{4}]$

Then apply Thm after pblm 19.

$f(0) < 0$

$f(\frac{\pi}{4}) > 0$ i.e. by IVT there exists at least one solution

Since $f'(x) > 0 \ \forall x$ in $(0, \frac{\pi}{4})$ there cannot be a 2nd zero i.e. there is exactly 1 zero.

Law of the mean.
ALSO called MEAN VALUE thm.

The law of the mean extends Rolle's thm by stating for any chord of a fctn there is a tangent with the same slope. Carefully stated!

INST RATE change = Ave RATE change somewhere.

Let f be a continuous fctn on the [a, b] & have a derivative in (a, b). Then ∃ at least one $\overline{X} \in (a,b)$ ∋

LAW of the mean ⟹ $f'(\overline{X}) = \dfrac{f(b)-f(a)}{b-a}$

Notice no LIMIT on RIGHT side!

Slope of f at $\overline{X}$ equals slope of chord fcn $(a, f(a)) \& (b, f(b))$ slope formula.

eg 1 $f(x) = 2x^3 - 8x + 1$ $[1, 3]$

f	2	0	−8	1
1	2	2	−6	−5
3	2	6	10	31

$f'(x) = 6x^2 - 8$

$\dfrac{f(b)-f(a)}{b-a} = \dfrac{31+5}{3-1} = \dfrac{36}{2} = 18$

Set $6x^2 - 8 = 18$

$6x^2 = 26$

$x^2 = \dfrac{13}{3}$

$x = \pm\sqrt{\dfrac{13}{3}}$ reject neg not in interval.

$\therefore \overline{X} = \sqrt{\dfrac{13}{3}}$ this is the $\overline{X}$ guaranteed by law of the mean.

eg 2 car speedometer 3 hr 210 miles! Ave speed was 70 then it speedometer must have read 70 mph!

more ↓

Translated to english, Average rate of change is, sometime during experiment, = to instantaneous rate of change.

Corollary 3 (to Law of the mean)

If f is continuous on $[a,b]$ & has a positive derivative on the open interval (a,b) then f is increasing on $[a,b]$, ie $f(b) > f(a)$

PROOF we have f is continuous & $f' > 0$ for this interval

The law of the mean says for some $\overline{X}$ where $a < \overline{X} < b$

$$f'(\overline{X}) = \frac{f(b) - f(a)}{(b-a)}$$

we know $f'(\overline{X}) > 0$ because f' is positive $\forall x$ in (a,b)

Also we know $b - a > 0$ since $a < b$.

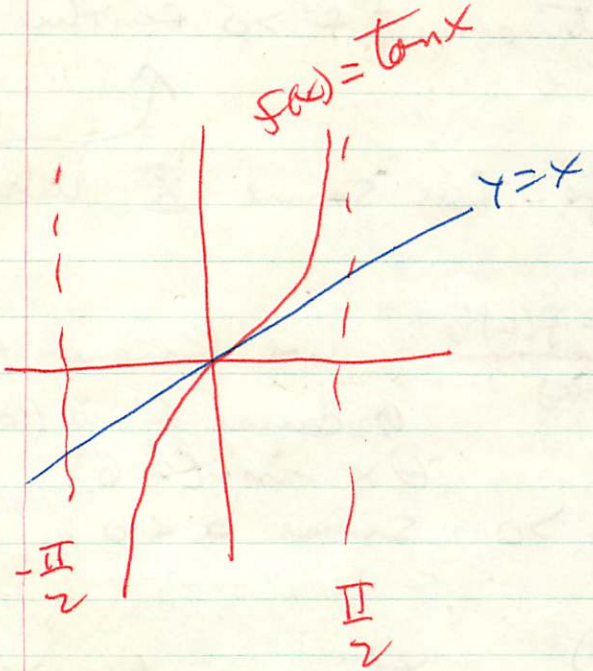
So we have

$$+ = \frac{(\ \ )}{+}$$

This implies that the Numerator Must be Positive ie

$f(b) - f(a) > 0$ Therefore

$f(b) > f(a)$ which is to say that as x goes from a to b $f(x)$ is increasing. Q.E.D.



is the blue line above or below the red one (or partially above etc)?

Summary:

Rolle's thm requires if $f(a)=f(b)$ then $\exists$ $\overline{x} \in (a,b) \Rightarrow f'(\overline{x})=0$.

Rolle's thm leads to the Mean value thm / Law of the mean which says that $f'(\overline{x}) = \frac{f(b)-f(a)}{b-a}$ for $\overline{x} \in (a,b)$

which leads to Corollary 3 (See proof on previous pg) which says.

if f is continuous and if $f' > 0$ on (a,b) then f is increasing.

So in order to Prove (for eg) $\tan x > x$, $0 < x < \frac{\pi}{2}$

we can define $f(x) = \tan x - x$

then $f'(x) = \sec^2 x - 1 = \tan^2 x$

so we know $f'(x) > 0$.

f is continuous (arithmetic combination of 2 continuous fctn is continuous)

so by Corollary 3 f is increasing.

that is to say $f(b) > f(a)$ where $a < b$.

Since $f(0) = 0$ then for any other $x > 0$ $f(x) > 0$ but $f(x) = \tan x - x$ so $\tan x - x > 0$ which is to say

$\tan x > x$

(it increases from zero to)

Caution:
The "reference pt" is needed to make the transition from "increasing" to Positive.
example follows.

WTP $e^x > 1 \ \forall x > 0$

Defin $f(x) = e^x - 1$ WTS $F > 0 \ \forall x > 0$

$f'(x) = e^x \ \therefore f'(x) > 0 \ \forall x$

Reference pt $f(0) = 1 - 1 = 0 \ \therefore$

$f(x) = e^x - 1 > 0 \ \forall x > 0$

ie f increases from zero

$\therefore f(x) = e^x - 1 > 0$

$e^x > 1$

end of lecture

eg 5 WTS $e^x > 1+x \quad \forall x > 0$

define $f(x) = e^x - x - 1$ Now we need to show $f(x) > 0 \ \forall x > 0$ so:

$$f'(x) = e^x - 1$$

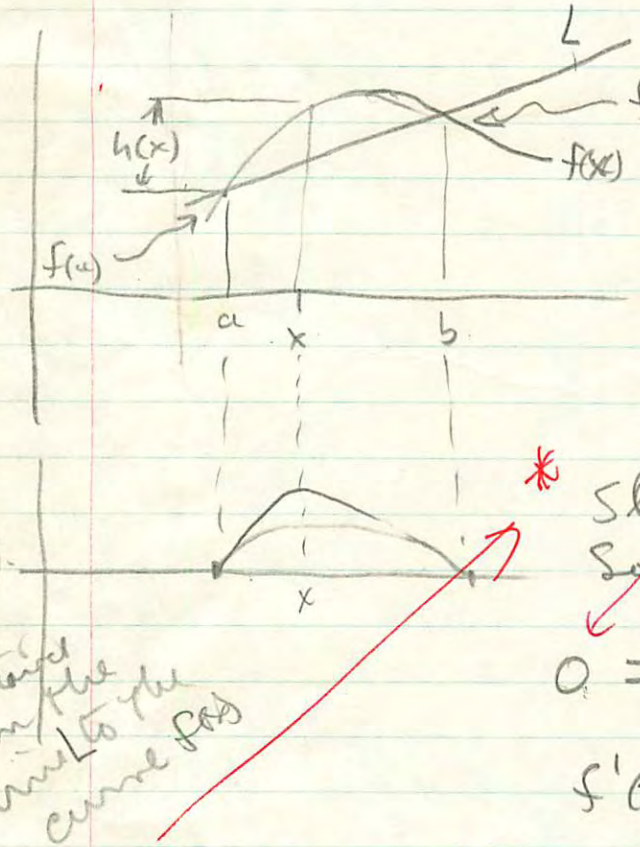
$\forall x > 0 \quad f'(x) > 0$ $\therefore$ f is an increasing fctn.

$f(0) = 0 \quad f(x) > f(0) \ \forall x > 0$ because f is increasing $\therefore$ $f(x)$ is positive $\forall x > 0$. so it is increasing from zero.

$$e^x - x - 1 > 0 \quad \forall x > 0$$

$$e^x > x + 1 \quad \forall x > 0$$

Proof of Law of the mean, also called MEAN VALUE Thm IN MANY other texts



$f(x)$ is fctn. $g(x) =$ Line.

$h(x) = f(x) - g(x)$

Noticed $h(a) = h(b) = 0$.

by Rolle's Thm $\exists \bar{x} \ni h'(\bar{x}) = 0$ solved

$$h'(\bar{x}) = f'(\bar{x}) - g'(\bar{x})$$

Slope of $g(x)$ i.e. Line: $\frac{f(b)-f(a)}{b-a}$

So we have

$$0 = f'(\bar{x}) - \frac{f(b)-f(a)}{b-a} \text{ i.e.}$$

$$f'(\bar{x}) = \frac{f(b)-f(a)}{(b-a)}$$ which is exactly the Law of the mean.

* remember $g(x)$ is just a straight line $\therefore$ its slope ANY value is $\frac{y_2-y_1}{x_2-x_1}$. Even at $x = \bar{x}$

6.2
P. 17

① $f(x) = x^3 - x^2$ $[-1, 2]$

	1	−1	0	0
−1	1	−2	2	−2
2	1	1	2	4

$3x^2 - 2x \overset{\text{Set}}{=} \frac{4+2}{2+1}$

$3x^2 - 2x - 2 = 0$

$x = \frac{2 \pm \sqrt{4+24}}{6}$

$x = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$ both good sol.

③ $f(x) = 5x$ $[1, 3]$ $f(1) = 5$ $f(3) = 15$

$5 \overset{\text{Set}}{=} \frac{15-5}{3-1}$

$10 = 10$ True $\forall x$

⑤ $f(x) = \cos x$ $[0, 4\pi]$ $\cos 0 = 1$ $\cos 4\pi = 1$

$-\sin x \overset{\text{Set}}{=} 0$

$x = \pi, 2\pi, 3\pi$

⑥ $f(x) = e^x$ $(2, 3)$

$e^x \overset{\text{Set}}{=} \frac{e^3 - e^2}{3-2}$ $\ln(e^3 - e^2)$

$e^x = e^2(e-1)$

$x = \ln e^2(e-1) = 2 + \ln(e-1)$

⑦ $f(x) = x \ln x$ $(1, e)$

$f'(x) = \ln x + 1$ $1 + \ln x \overset{\text{Set}}{=} \frac{e}{e-1}$

$\ln x = \frac{e}{e-1} - 1 = \frac{e - e + 1}{e-1}$

$\ln x = \frac{1}{e-1}$

$x = e^{\frac{1}{e-1}}$

(8) There is some point in the interval where the density is equal to the average density.

(9) There is some point where the magnification is equal to the average magnification for the interval.

(10) a) $\sqrt{1+x^5}$ $\quad \frac{5x^4}{2\sqrt{1+x^5}}$

b) $\frac{\cos^2 3x}{\sin 2x}$ $\quad \frac{-6\sin 2x \cos 3x \sin 3x + 2\cos^2 3x \cos 2x}{\sin^2 2x}$

c) $e^{-x}\sec 3x$ $\quad -e^{-x}\sec 3x + 3e^{-x}\sec 3x \tan 3x$
$e^{-x}\sec 3x\,(3\tan 3x - 1)$

d) $\sin^{-1} x^3$ $\quad \frac{3x^2}{\sqrt{1-x^6}}$

e) $\ln|\sec 2x|$ $\quad \frac{2|\sec 2x \tan 2x|}{|\sec 2x|} = 2|\tan 2x|$

(11) $f' = 2$ $\quad \frac{17-7}{8-3} = 2$
for some x $\quad 3 < x < 8$

(12) a) Corollary 2 Same velocity. constant diff. between curves ie $f(x) = 5$ $\quad g(x) = 5 + c$.

b) corollary 1

(13) a) $\ln 3x$ $\quad \ln 2x$ $\quad$ b) $\ln 3x - \ln 2x = C$
$\frac{3}{3x} = \frac{1}{x}$ $\quad \frac{2}{2x} = \frac{1}{x}$ $\quad \ln\frac{3}{2} = C$

(16) See lecture below

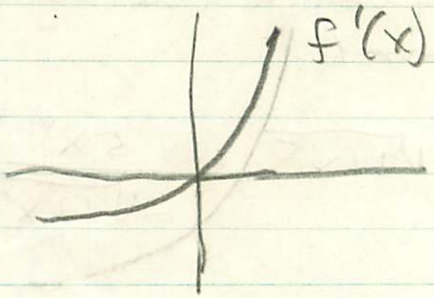
WTP $e^x > 1+x \quad \forall x<0$ ie $e^x - x - 1 > 0$

define $f(x) = e^x - x - 1$ WTP $f(x) > 0 \quad \forall x<0$

$f'(x) = e^x - 1$

$f(0) = 1 - 0 - 1 = 0$

$f'(x) < 0 \quad \forall x < 0$



$\forall x<0 \quad f(x)$

(16) using how of the mean

(15)

$f(a) = g(a)$

$f'(x) \leq g'(x)$

$H = f(x) - g(x)$

$H(a) = 0$

$H'(x) = f'(x) - g'(x) \leq 0$

So H decreases from a $\therefore H \leq 0$

So $f(x) - g(x) \leq 0$

$f(x) \leq g(x)$

$f(b) \leq g(b)$ Q.E.D.

14 a) $\cos^2 x \qquad -\sin^2 x$

$-2\cos x \sin x \qquad -2\sin x \cos x$

b) $\cos^2 x + \sin^2 x = c \qquad c = 1$

(15) $g(x) = y =$ equation of L.

Slope of $g(x) = \dfrac{f(b)-f(a)}{b-a}$ through the pt $(a, f(a))$

$y = g(x) - f(a) = \dfrac{f(b)-f(a)}{b-a}(x-a)$

a) $g(x) = f(a) + \left(\dfrac{x-a}{b-a}\right)(f(b)-f(a))$

b) $g'(x) = f'(a) + \dfrac{1}{(b-a)} f(b)-f(a)$

(16) Define $h(x) = e^x - x - 1 \quad \forall x < 0$

wts $h(x) < 0 \quad \forall x < 0.$

See Lecture Next 3 pgs.

$h'(x) = e^x - 1 \quad \forall x < 0 \quad e^x - 1 < 0 \therefore$

h is a decreasing fctn.

$h(0) = 0$ So $\forall$ other $x \quad h(x) < h(0)$

$h(x) < 0 \therefore e^x - x - 1 < 0$

(17) $f(a) = g(a) \qquad f'(x) < g'(x)$

$h'(x) = g'(x) - f'(x) > 0$

$h(x) = g(x) - f(x)$

wts $h(x)$ is increasing.

then

$h(a) = 0 \qquad h(x) > h(a) \quad \forall x > a$

$g(x) - f(x) > 0.$

$g(x) > f(x).$

are these same the x?

is $h' > 0$? even if f & $g < 0$?

yes.

OR

$g'(x) = \dfrac{g(b)-g(a)}{b-a} \qquad f'(x) = \dfrac{f(b)-f(a)}{b-a}$ by Mean Value Thm

$\dfrac{g(b)-g(a)}{b-a} > \dfrac{f(b)-f(a)}{b-a}$ given $f' > g'$

$g(b)-g(a) > f(b)-f(a)$ since $f(a) = g(a)$

$g(b) > f(b).$

Problem 16 Lecture Pg 1.7

Pg 1 of Lecture.

"I tried to do this problem this morning and I suddenly found that I had done it wrong. It seems that I proved it was untrue but I also can show the problem is true soooo WE have a problem.

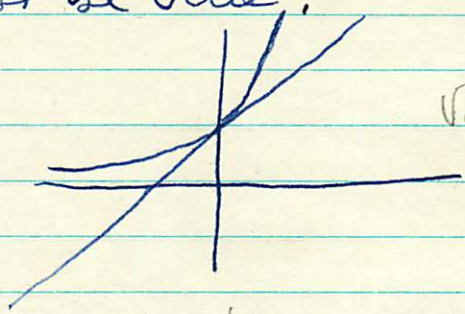
WTP $e^x > 1+x \quad \forall x<0$ how do we start?

Sooner or Later Students will consider these intervals – if it takes longer than 5 minutes "Let me show you that it must be true.

if $x \le -1$ we get $+ > -$ obvious

if $-1 < x < 0$ is the only area where we have trouble.

Draw $Y = e^x$
$Y = x+1$



$\sqrt{e} < 2$
$e < 4$

we know $e \approx 2.718$

Consider them

$x = -\frac{1}{2}$ to get $\frac{1}{e^{\frac{1}{2}}} \quad \frac{1}{e^{\frac{1}{2}}} > 1 - \frac{1}{2}$

$\frac{1}{e^{\frac{1}{2}}} > \frac{1}{2}$ ie $\frac{1}{\sqrt{e}} > \frac{1}{2}$

$.6065 > .5$

$x = -2 \quad \frac{1}{e^2} > 1-2$ ie $.135 > -1$

$x = -10 \quad \frac{1}{e^{10}} > 1-10$ ie $.000045 > -9$

This should do it. If they still do not talk about intervals tell them – and then out because this is Not an entertainment class.

Pg 177 #16

"So far we have considering the interval $(-1, 0)$. Lets Say $[-1, 0]$ why? Law of the mean has a closed interval in the IF statement

Lets Define a fctn. $e^x > 1+x \Rightarrow e^x - x - 1 > 0$

Define $f(x) = e^x - x - 1$ WTS $f(x) > 0$

write these in this order →

$f(0) = 0$

$f(-1) = \frac{1}{e}$

clearly, $\frac{1}{e} > 0$ so $e^{x}+1 < 0$ in this interval.

This will Disguise Solution for a while.

$f'(x) = e^x - 1$

Negative Exponent

for $\forall x \ni -1 < x < 0$ $f'(x) < 0$

that is f is a Decreasing fctn!

So what?"

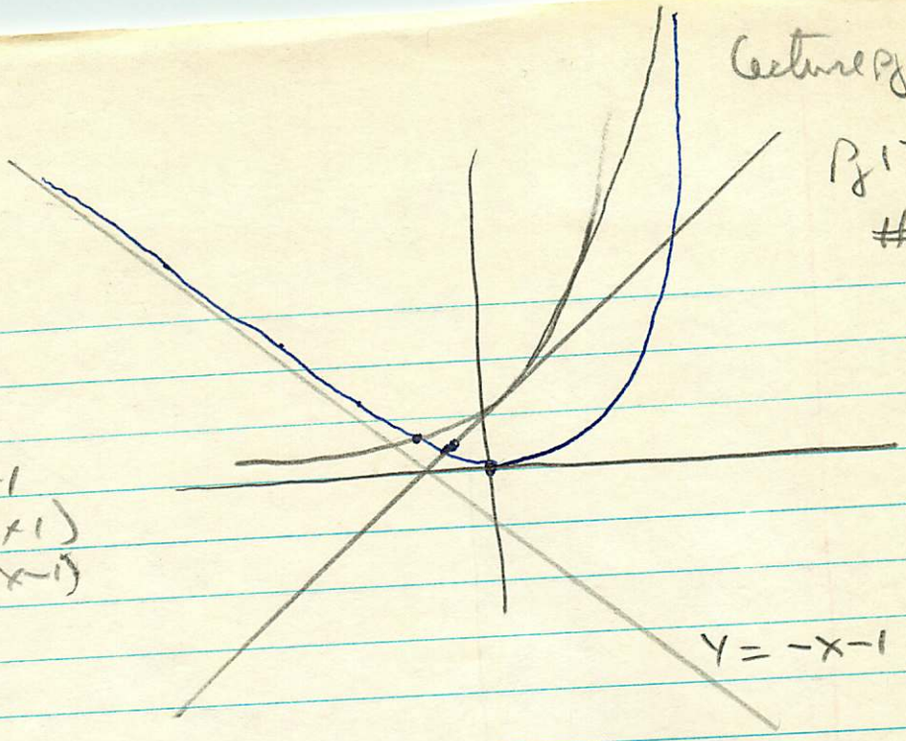
Students should argue that this is not what we want to show. Let them go off on a tangent for 5-10 minute. Use the time to re establish where everything came from.

"But f is a decreasing fctn! — Decreasing from where? (from $\frac{1}{e}$ to zero) So what? So f is indeed positive — it is Decreasing true but decreasing to zero implies it must be Positive so it can decrease TO zero.

Conclusion; $f > 0$ so $e^x - x - 1 > 0$ so $e^x > x + 1$ QED Notice that we are using the thm proved Yesterday.

↓

Pg 177
#16



Lets look at the graph.

Since $f(x) = e^x - x - 1$
$= e^x - (x+1)$
$= e^x + (-x-1)$

we just add range values.

as $x \to -\infty$ $e^x \to 0$ and we approach $y = -x-1$
as $x \to +\infty$ e^x much larger than $x-1$ $\therefore$ we approach e^x

In K is graph of f & it is indeed ≥ 0 $\forall x \in \mathbb{R}$.

Now show pblm 17

~~(Flips back)~~

#17 $f(a) = g(a)$ $f'(x) < g'(x)$ w.t.p. $f(b) < g(b)$

Let $h(x) = f(x) - g(x)$
$h(a) = f(a) - g(a) = 0$
$h'(x) = f'(x) - g'(x) < 0$ h is a decreasing fctn

as x goes from a to b, h decreases from $h(a) = 0$ thus

$h(x) < 0$ so
$f(x) - g(x) < 0$
$f(b) - g(b) < 0$
$f(b) < g(b)$ ∎

(18) define $h(x) = \tan x - x$ wts $h(x) > 0 \ \forall x > 0$

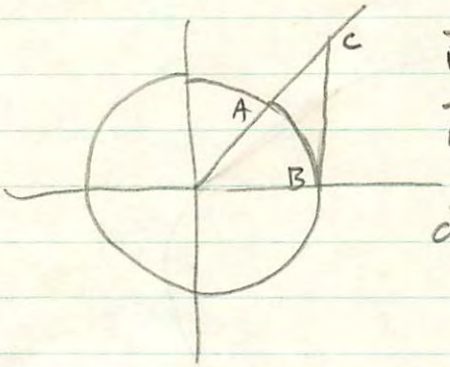
$h(0) = 0$ $h'(x) = \sec^2 x - 1 > 0$ since $\sec x > 1 \ \forall x > 0$ $\therefore$ increasing.

$\therefore h(x) > h(0) \ \forall x > 0$

$\tan x - x > 0$

a) $\tan x > x$.

b)



$\overline{BC} = \tan x$ (Pg 83)

$\overset{\frown}{AB} = x$

a) above says $\overline{BC} > AB$

(19) $\dfrac{x}{1+x^2} < \tan^{-1} x < x \quad \forall x > 0$.

$f(x) = \dfrac{x}{1+x^2}$ $g(x) = \tan^{-1} x$ $h(x) = x$

$f'(x) = \dfrac{1+x^2-2x^2}{(1+x^2)^2}$ $g'(x) = \dfrac{1}{x^2+1}$ $h'(x) = 1$

$= \dfrac{1-x^2}{(1+x^2)^2}$

Note: $f(0) = g(0) = h(0)$. Now we need to prove $\forall x > 0$ see

Need to show $f'(x) < g'(x) < h'(x)$ then $f(b) < g(b) < h(b)$

$\dfrac{1-x^2}{(1+x^2)^2} \overset{?}{<} \dfrac{1}{x^2+1}$

$1 - x^2 \overset{?}{<} x^2 + 1$

$1 - 0 < 2x^2$

True $\forall x$

$\dfrac{1}{x^2+1} \overset{?}{<} 1$

$1 \overset{?}{<} x^2 + 1$

$0 < x^2$

true $\forall x$ $\therefore$ by Pblm 17 above,

$f(b) < g(b) < h(b) \Rightarrow$

$\dfrac{x}{1+x^2} < \tan^{-1} x < x$

Pg 178

20 WTS $1-x+x^2-x^3 < \frac{1}{1+x} < 1-x+x^2-x^3+x^4$

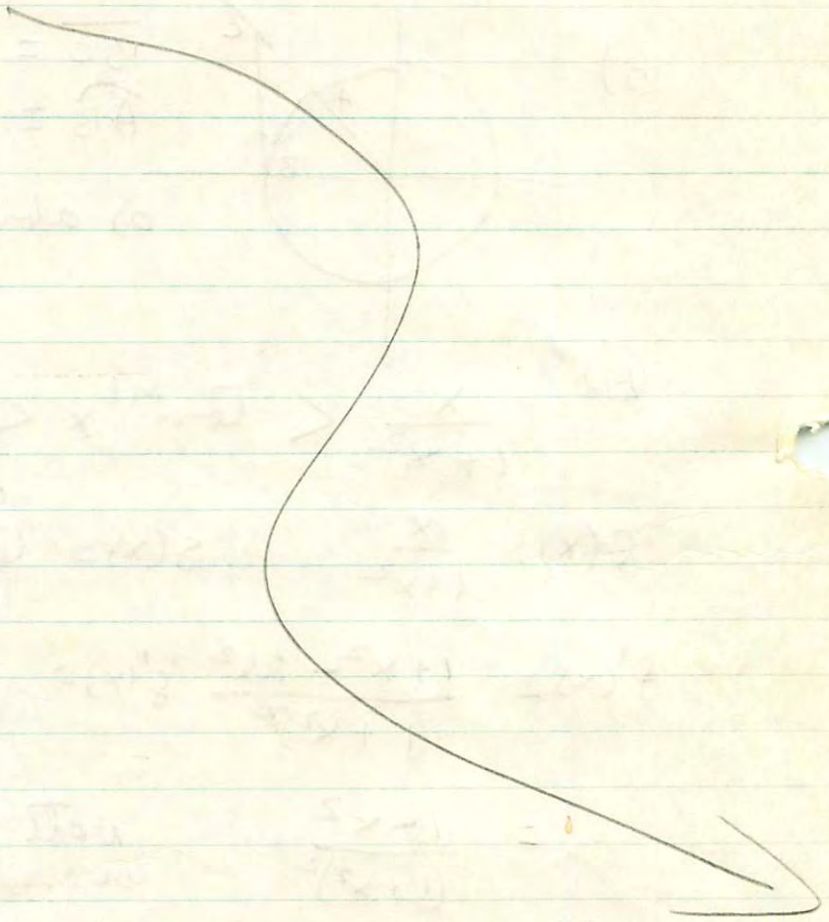
$(1+x) \Rightarrow$

$1-x^4 < 1 < 1+x^5$

for $x > 0$

obviously true

again obviously true,



20 $x \in (0,1)$ WTS $1-x+x^2-x^3 < \frac{1}{1+x} < 1-x+x^2-x^3+x^4$

a) add $-1+x-x^2+x^3$ to all 3,

$$0 < \frac{1}{1+x} - 1 + x - x^2 + x^3 < x^4$$

$$\begin{array}{r|l} & 1-x+x^2-x^3 \\ \hline 1+x & 1 \\ & 1+x \\ \hline & -x \\ & -x-x^2 \\ \hline & x^2 \\ & x^2+x^3 \\ \hline & -x^3 \\ & -x^3-x^4 \\ \hline & x^4 \end{array}$$

$$0 < 1-x+x^2-x^3 + \frac{x^4}{1+x} - 1 + x - x^2 + x^3 < x^4$$

$$0 < \frac{x^4}{1+x} < x^4$$

obviously true. $\forall x > 0$

$$\frac{x^4}{1+x} < x^4$$

$$\frac{1}{1+x} < 1$$

$$1 < 1+x$$

$$0 < x \quad \text{True } \forall x > 0$$

b) we have shown $f'(a) < g'(a) < h'(a)$

$f(0) = g(0) = h(0)$ i.e. by Pblm 17 $f < g < h$

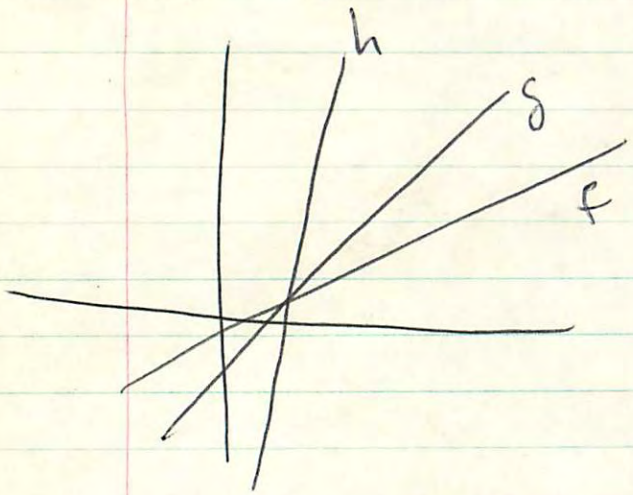
where $f = 1-x+x^2-x^3$ etc.

c) $\ln 1.2 = \ln\left(1+\frac{1}{5}\right)$

$$\frac{1}{5} - \frac{1}{50} + \frac{1}{375} - \frac{1}{2500} < \ln 1.2 < \ldots + \frac{1}{15625}$$

$$.182266 < .18232 < .18233$$

a general Summary of Problem 22 & 23
as well as most of the problems
in this Section is;



f, g, h have a common pt call it $x = a$
$\forall x > a \quad f < g < h$ since each ~~exists~~ increases faster than the fctn before it.

(21) $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{N+1} x^N}{N} < \ln(1+x) <$

$x - \frac{x^2}{2} + \cdots \frac{(-1)^{N+2} x^{N+1}}{N+1}$

(22) $0 < \cos x < 1 \quad (0, \frac{\pi}{2})$ wts $0 < \sin x < x$

a) let $f'(x) = 0 \quad g'(x) = \cos x \quad h'(x) = 1$

$f' < g' < h'$

$f(x) = 0 \quad g(x) = \sin x \quad h(x) = x$

$f(0) = g(0) = h(0)$ ∴ by thm 17

$f(x) < g(x) < h(x) \Rightarrow 0 < \sin x < x.$

b) wts $-1 < -\cos x < -1 + \frac{x^2}{2}$

let $f' = 0 \quad g'(x) = \sin x \quad h'(x) = x$

$f = -1 \quad g(x) = -\cos \quad h = -1 + \frac{x^2}{2}$

Now $f(0) = g(0) = h(0)$. so

$-1 < -\cos x < -1 + \frac{x^2}{2}$ ★

c) $-x < -\sin x < -x + \frac{x^3}{2 \cdot 3}$

from ★ $f' = -1 \quad g'(x) = -\cos x \quad h'(x) = -1 + \frac{x^2}{2}$ so

$f(x) = -x \quad g(x) = -\sin x \quad h(x) = -x + \frac{x^3}{2 \cdot 3}$

etc,

each of the others comes in the same form.

(23) a) 22 continues. for Power 6 cos is >0.

b) for Power 7

$$-1 < -\cos x < -1 + \frac{x^2}{2}$$

$$-x < -\sin x < -x + \frac{x^3}{2\cdot 3}$$

$$1 - \frac{x^2}{2} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{2\cdot 3\cdot 4}$$

$$x - \frac{x^3}{2\cdot 3} < \sin x < x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5}$$

23 a) $\Rightarrow$ $$-1 + \frac{x^2}{2} - \frac{x^4}{2\cdot 3\cdot 4} < -\cos x < -1 + \frac{x^2}{2} - \frac{x^4}{2\cdot 3\cdot 4} + \frac{x^6}{2\cdot 3\cdot 4\cdot 5\cdot 6}$$

multiply by -1 to obtain (23 a)

23 b) $\Rightarrow$ $$-x + \frac{x^3}{3} - \frac{x^5}{2\cdot 3\cdot 4\cdot 5} < -\sin x < -x + \frac{x^3}{2\cdot 3} - \frac{x^5}{2\cdot 3\cdot 4\cdot 5} + \frac{x^7}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

If use Product rule
$x \cdot 1 + x \cdot 1$
$2x$

① $f(x) = x^2$ $\quad g(x) = x \cdot x$

$3^2 = \underbrace{3+3+3}_{3}$ $\quad 4^2 = \underbrace{4+4+4+4}$

② $g(x) = \underbrace{x+x+x+\cdots+x}_{x \text{ times}}$ $\quad 5^2 = 5+5+5+5+5$

$5(1+1+1+1+1)$

$5 \cdot 5$

This → is only valid if x is a positive integer.

③ If $f(x) = g(x)$ Then $f'(x) = g'(x)$ obviously true

↳ $\sin^2 x = 1 - \cos^2 x$

$2\sin x \cos x = -2\cos x(-\sin x)$

④ $x^2 = \underbrace{(x+x+x+\cdots+x)}_{x \text{ terms}}$

⑤ $2x = \underbrace{1+1+1+\cdots+1}_{x \text{ terms}}$

⑥ $2x = x$ x is a number of terms > 0 $\therefore x \neq 0$

⑦ $\div x$ yields $2 = 1$

6.3

pg 178–182

Thm 1 $e^x > 1 + x + \frac{x^2}{2} \Rightarrow e^x > \frac{x^2}{2}$

Proof follows those of Previous section.

Thm 2 $\lim_{x\to\infty} \frac{x}{e^x} = 0$ i.e. $e^x > x$ as $x \to \infty$

e^x gets larger FASTER.

also $\lim_{x\to\infty} \frac{e^x}{x} = \infty$

Thm 3

$\lim_{x\to\infty} \frac{\ln x}{x} = 0$ i.e. $\ln x$ gets

Larger as $x \to \infty$ but slower than x.

Thm 4 $\lim_{x\to\infty} \frac{b^x}{x^a} = \infty$ i.e. exponential grows faster than Power Series.

Slowest ↓ FASTEST

constant — No growth

$\ln$

x^{constant}

$(\text{constant})^{\text{variable}}$

Variable

Finally

$\ln x < x^a < b^x$

① $\lim_{x\to\infty} \frac{x^3}{e^x} = 0$ ② $\lim_{x\to\infty} x^5 e^{-x} = 0$

③ $\lim_{x\to-\infty} x^5 e^{-x} \Rightarrow \infty$ ④ $\lim_{x\to-\infty} x^4 e^{-x} \Rightarrow \infty$

⑤ $\lim_{x\to\infty} \frac{(1.01)^x}{x^{50}} \Rightarrow \infty$ ⑥ $\lim_{x\to\infty} \frac{(\ln x)^2}{2^x} = 0$

⑦ $\lim_{x\to\infty} \frac{\ln x}{x^3} \Rightarrow 0$ ⑧ $\lim_{x\to0^+} \frac{\ln x}{x^3} = \lim_{a\to\infty} \frac{\ln\frac{1}{a}}{(\frac{1}{a})^3}$

$\Rightarrow \lim_{a\to\infty} \frac{-\ln a}{\frac{1}{a}} = \lim_{a\to\infty} -a\ln a$

$-\infty$

⑨ $\lim_{x\to\infty} \frac{\log_{10} x}{x} = 0$

⑩ $\lim_{x\to\infty} \frac{2^x}{e^x} \Rightarrow 0$ ⑪ $\lim_{x\to\infty} \frac{\log_2 x}{\log_3 x} = \frac{\frac{\ln x}{\ln 2}}{\frac{\ln x}{\ln 3}} = \frac{\ln 3}{\ln 2}$

⑫ $\lim_{x\to+\infty} 2^x e^{-x}$

$\lim_{a\to\infty} \frac{e^a}{2^a} = +\infty.$

⑬ $\lim_{x\to0^+} x\ln x$ let $t = \frac{1}{x}$

$\lim_{t\to+\infty} \frac{\ln\frac{1}{t}}{t} = 0$

⑭ $\lim_{x\to\infty} x^{\frac{1}{x}}$ $\lim_{x\to\infty} \ln x^{\frac{1}{x}} = 0$ (see 13)

$y = x^{\frac{1}{x}}$

$\ln y = \ln x^{\frac{1}{x}}$ → imply $\lim_{x\to0} \ln y = 0$

$\ln y = 0$ when $y = 1$

Since $\ln x^{\frac{1}{x}} \Rightarrow 0$

$x^{\frac{1}{x}} \Rightarrow 1$ $\therefore \lim_{x\to\infty} x^{\frac{1}{x}} \Rightarrow 1$

does this mean $\infty^0 = 1$?

(5) a) $\frac{\log_{10} x}{x}$ $\quad 10^2 \mid \frac{2}{10^2}$ $\quad 10^3 \mid \frac{3}{1000}$ $\quad 10^4 \mid \frac{4}{10000}$

b) $\lim_{x \to \infty} \frac{\log_{10} x}{x} \Rightarrow 0$

(6) a) $\frac{\sqrt{3x^2-1}}{e^{5x}}$ $\quad \frac{\frac{1}{2}(6x)}{\sqrt{3x^2-1}}$

$\frac{1}{e^{10x}}\left(\frac{3x e^{5x}}{\sqrt{3x^2-1}} - 5e^{5x}\sqrt{3x^2-1}\right) = \frac{3x - 15x^2 + 5}{e^{5x}\sqrt{3x^2-1}}$

b) $\sec\sqrt{x}$ $\quad \frac{\sec\sqrt{x}\tan\sqrt{x}}{2\sqrt{x}}$

c) $(\cos 5x)^3$ $\quad -15\cos^2 5x \sin 5x$

D) $\ln\sqrt[3]{(1+\tan 2x)^2} = \frac{2}{3}\ln(1+\tan 2x)$

$\frac{2 \cdot 2\sec^2 2x}{3(1+\tan 2x)}$ $\quad \frac{4}{3}$

e) $\sqrt{x}\tan^{-1} 2x$

$\frac{\tan^{-1} 2y}{2\sqrt{x}} + \frac{2\sqrt{x}}{1+4x^2}$

f) $x^3 \sin^{-1} e^{2x}$

$3x^2 \sin^{-1} e^{2x} + \frac{2x^3 e^{2x}}{\sqrt{1-e^{4x}}}$

(17) a) $\frac{1}{x^4}$ $\frac{-x^{-3}}{3}$ b) $x^{\frac{1}{3}}$ $\frac{3x^{4/3}}{4}$

c) $\frac{3}{x}$ $3\ln x$ d) $x^3 - 2x^2$ $\frac{x^4}{4} - \frac{2}{3}x^3$

e) $\frac{5}{1+x^2}$ $5\tan^{-1}x$ f) $\frac{3x}{1+x^2}$ $\frac{3}{2}\ln(1+x^2)$

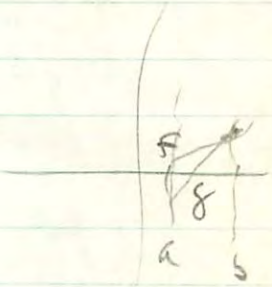
g) $\frac{1}{(1+3x)^2}$ $\frac{-1}{3(1+3x)}$ h) $\frac{1}{1+3x}$ $\frac{1}{3}\ln(1+3x)$

i) $\frac{1}{\sqrt{1-4x^2}}$ $\frac{1}{2}\sin^{-1}2x$

(18) $f(b) = g(b)$ (a,b) $f'(x) < g'(x)$

$f'(x) - g'(x) < 0$

WTP $f(a) > f(b)$



① $f(a) \neq f(b)$ shapes are different.

② ~~assume $f(a) < g(a)$~~

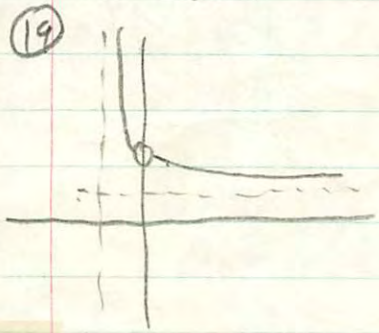
③ Let $h(x) = f(x) - g(x)$ want to show $h(x)$ is decreasing. ie the difference between the two fcns is decreasing.

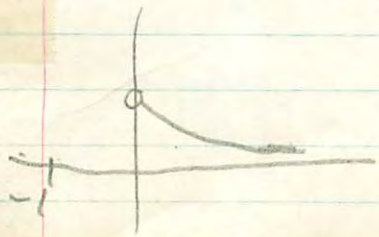
$h'(x) = f'(x) - g(x) < 0$

$h(x) \geq h(b) = 0$

$f(x) - g(x) > 0$ ∴ $f(x) > g(x)$.

Pg 26 (19)





$f(x) = (1+x)^{\frac{1}{x}}$ $\ln f = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x)$

$\lim_{x\to\infty} \frac{\ln(1+x)}{x} = 0.$

$\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{1}{x}\ln(1+x) =$

$\lim_{x\to 0} \ln(1+x)^{\frac{1}{x}} = 1$

$\lim_{x\to -1} \ln(1+x)^{\frac{1}{x}}$

Start here.

Question: If $\ln(f(x))$ is decreasing $\forall x \in \mathbb{R}$, what can be said about f?

Consider: $y = \ln f$ Given y is decreasing.

$y' = \frac{1}{f} f'(x)$ Since y decreasing $y' < 0$

$\frac{1}{f} f'$ ~~is decreasing~~ < 0

If $f > 0$ Then $f' < 0$ and f is also a decreasing fct $\forall x \in \mathbb{R}$!

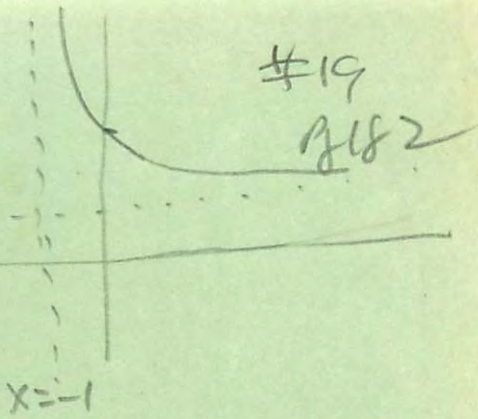
or if $f < 0$ $f' > 0$

~~Right.~~ f must be > 0

Because Domain of $\ln$ is $\ln(f(x))$

#19
§1.8 #2

$f(x) = (1+x)^{\frac{1}{x}}$; $f(0) = e$



Core I $x>0$ let $g(x) = \ln f(x)$

Justifications on previous page

$g(x) = \ln(1+x)^{\frac{1}{x}}$

WTP $g' < 0$.

$g(x) = \frac{1}{x}\ln(1+x)$ (u, v)

$g' = \frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)}$

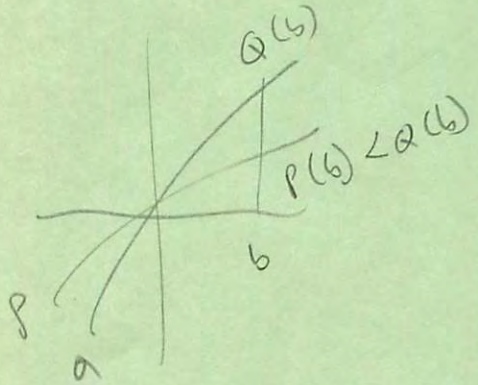
$= \frac{x - (1+x)\ln(1+x)}{x^2(1+x)} \overset{?}{<} 0$

bottom always positive so

$x < (1+x)\ln(1+x)$ (u, v)

$P(x)$ $Q(x)$

$P(0) = Q(0) = 0$



$1 < \ln(1+x) + \frac{(1+x)}{(1+x)}\frac{1}{}$

$1 < \ln(1+x) + 1$

SINCE $x>0$ $0 < \ln(1+x)$ True: $\ln(1) = 0$

for all $x>0$: $P' < Q'$ and

Corollary 3. $P(0) = q(0) = 0$. so $P(x) < q(x)$.

So for $x>0$ g is a decreasing fctn

$\therefore$ $f(x)$ is also a decreasing fctn.

$f(x) = (1+x)^{\frac{1}{x}}$ $\quad f(0) = e \quad -1 < x < 0$

WTS f is decreasing fctn.

Let $g(x) = \ln(1+x)^{\frac{1}{x}} = \frac{1}{x}\ln(1+x)$ (u v)

$$g'(x) = \frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)} \overset{?}{<} 0$$

$$? - \longrightarrow \quad \frac{-(1+x)\ln(1+x) + x}{x^2(1+x)}$$

$$+ \longrightarrow$$

$\overset{?}{\leftarrow}$ Really don't want this however, since I am using $x=0$ is right end of interval & really want g' greater than p'!

$\bigstar \quad x - (1+x)\ln(1+x) \overset{?}{<} 0$ (u v)

$$x < (1+x)\ln(1+x)$$

$p \qquad q$

$p(0) = 0 \qquad q(0) = 0$

$p' \qquad q'$

$$1 < \ln(1+x) + \frac{1+x}{1+x}$$

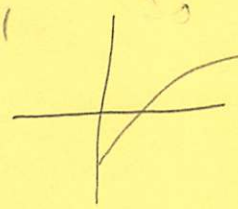
$$\ln(1+x) + 1$$

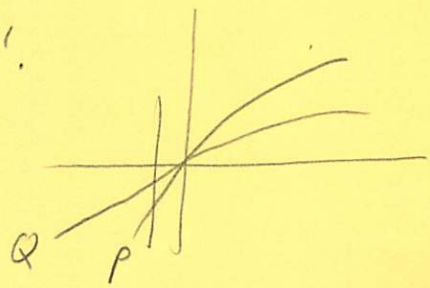
$$0 < \ln(1+x)$$

This $\Rightarrow$ not true

$\ln(1+x) < 0$

So $q' < p'$.





for $x < 0$ $Q > P$ which is EXACTLY what I wanted in $\bigstar$

$f(x) = (1+x)^{\frac{1}{x}}$ $f(0) = e$ $-1 < x < 0$

Pg 182 Prblm 19

wts f is decreasing fctn ie $\cap$

let $g(x) = \ln(f(x)) = \frac{1}{x}\ln(1+x)$

wts $g(x)$ is a decreasing fctn.

$$g'(x) = \frac{-\ln(1+x)}{x^2} + \frac{1}{x(1+x)}$$

$$= \frac{-(x+1)\ln(1+x) + x}{x^2(1+x)}$$

question: is $g'(x) < 0$? bottom is still positive.

So, is numerator < 0? if numerator < 0 then $g'(x) < 0$ and g is a decreasing fctn.

$$-(x+1)\ln(1+x) + x \overset{?}{<} 0$$

Let $h(x) = x - (x+1)\ln(1+x)$

$h(0) = 0$

$h'(x) = 1 - (\ln(1+x) + 1)$

$= -\ln(1+x)$

for $-1 < x < 0$, $\ln(1+x) < 0 \therefore h'(x) > 0$

h increases to zero $\therefore h(x) < 0$.

So $g'(x) < 0$ then g is a decreasing fctn.

Then f is also a decreasing fctn when $0 < x < 1$

$$x - (1+x)\ln(1+x) < 0$$

Let $h = x - (1+x)\ln(1+x)$

$h(0) = 0$

$h'(x) = 1 - (\ln(1+x) + 1)$

$= -\ln(1+x)$

$h'(x) < 0$

h decreases from zero.

$h < 0.$

$f(x) = (1+x)^{\frac{1}{x}}$ $f(0) = e$ $-1 < x < 0$

WTS f is dec fctn. ie show $g' < 0$

Define $g(x) = \ln(1+x)^{\frac{1}{x}}$

$u \quad v$

$g(x) = \frac{1}{x}\ln(1+x)$

$g'(x) = \ln(1+x)\left(\frac{1}{x^2}\right) + \frac{1}{x(1+x)}$

$= \frac{x - (x+1)\ln(1+x)}{x^2(x+1)}$

the denominator is > 0

$x - (x+1)\ln(1+x) \overset{?}{<} 0$

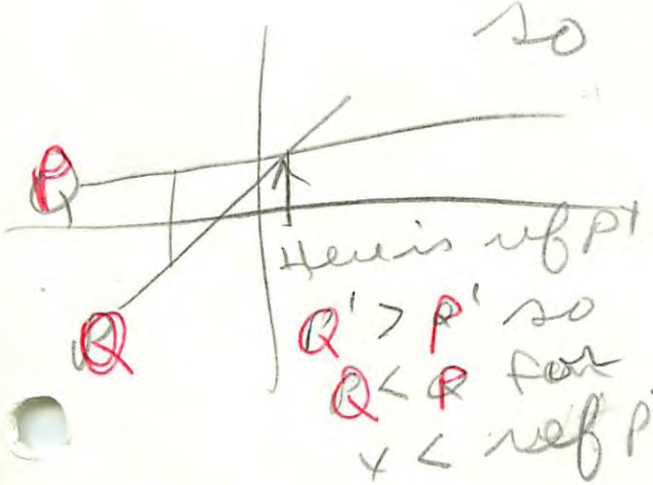
Let $P = x$ $Q = (x+1)\ln(1+x)$

$P(0) = Q(0) = 0$

$P' = 1$ $Q' = \ln(1+x) + 1$

for x $-1 < x < 0$ $\ln(1+x) < 0$

so $P' > Q'$



so $\forall x < 0$ $P < Q$

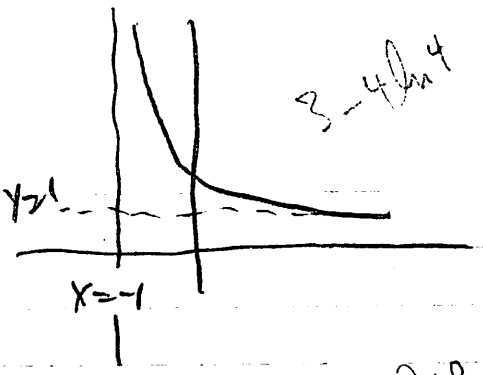
since $P < Q$

$x - (x+1)\ln(x+1) < 0$

and $g'(x) < 0$.

so f is dec fctn

in $-1 < x < 0$



3-4 hw 4

(19) $f(x) = (1+x)^{\frac{1}{x}}$; $f(0) = e$

Case I if $x > 0$

$f(0) = e$

1-2.10

let $g = \ln(1+x)^{\frac{1}{x}} = \frac{\ln(1+x)}{x}$

$g(0) \stackrel{?}{=} 1$ $\quad g'(x) = \frac{\frac{x}{1+x} - \ln(1+x)}{x^2} = \frac{x - (1+x)\ln(1+x)}{x^2(1+x)}$

as $x \to \infty$

$g'(x) \Rightarrow 0$ $\therefore$ g is decreasing to 0 this means that f is going to 1 $\therefore$ since f was at e and is going to 1 f is decreasing $\forall x > 0$

Case II $-1 < x < 0$

Let $g = \ln(1+x)^{\frac{1}{x}}$

$\lim_{x \to -1^+} \ln(1+x)^{\frac{1}{x}} = \lim_{x \to -1} \frac{\ln(1+x)}{x} = \frac{-\infty}{-1}$

The denominator is clearly a negative

The numerator goes to $-\infty$

$\therefore \lim_{x \to -1} \ln(1+x)^{\frac{1}{x}} \Rightarrow +\infty$

this implies $(1+x)^{\frac{1}{x}} \Rightarrow +\infty$ as $x \to -1$

Questions: 15

$g'(x) = \frac{x - (1+x)\ln(1+x)}{x^2(1+x)}$

$\ln \lim_{x \to 0} (1+x)^{\frac{1}{x}} = \ln e = 1$

$\therefore$ g goes from $+\infty$ to e as x goes fm -1 to 0

$\therefore$ g is decreasing $\forall x \ni -1 < x < 0$

WTS $f' < 0$; what would this mean about g?

See

$(1+x)^{\frac{1}{x}} > 0 \ \forall x \in (-1, 0)$ because $(1+x) > 0$

g is decreasing so f is also a decreasing fctn.

$a = -b$

20) $\lim_{x\to\infty} \frac{\ln x}{x^a}$ $a<0$ $= \lim_{x\to\infty} x^b \ln x$ $b>0$

$= \infty$

21) $a>0$

$\lim_{x\to\infty} \frac{\ln x}{x^a}$ let $c = x^a$

(22)

(23) $f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2} < 0 \quad \forall x \neq 0$

If $x_1 > x_2$ Then $f(x_1) \overset{?}{>} f(x_2)$ NO

$x_2 < x_1$

F is decreasing $\forall x$ f decreases from $(x_2, f(x_2))$ to $(x_1, f(x_1))$

But what if $x_2 < 0 < x_1$??

Example

Determine Asympotes & Sketch $f(x) = \dfrac{x^2-3}{x+2}$ ~~$\dfrac{x^3-1}{x+2}$~~

$x+2 \overset{\text{set}}{=} 0$

$x=-2$ equation of vertical asympote, also called "TILTED"

there may also be a ~~linear~~ horizontal or oblique asympote

Divide Numerator by denominator ie

$$x+2 \,\big|\, x^2-1 \qquad = \quad x-2+\frac{3}{x+2}$$

x^2+2x

$-2x$

$-2x-4$

3

as x gets large the $\frac{3}{x+2}$ becomes insignificant.
This curve approaches $y=x-2$

OR:

$$\begin{array}{r|rrr} & 1 & 0 & -1 \\ -2 & & -2 & 3 \\ \hline & 1 & -2 & \end{array} \Rightarrow$$

$y = x-2$ is equation of asympote.

$$f'(x) = \frac{2x(x+2) - x^2+1}{x^2+4x+4} = \frac{x^2+4x+1}{(x+2)^2}$$

$f' >$ when x^2+4x+1 is >0 (since denom is always >0)

$$x = \frac{-4 \pm \sqrt{16-4}}{2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3}$$

So. $f'=0$ when $x = -2 \pm \sqrt{3}$ consider x Large

x^2+4x+1 is then >0 so: $+ \;\; 0 \;\; - \;\; 0 \;\; +$

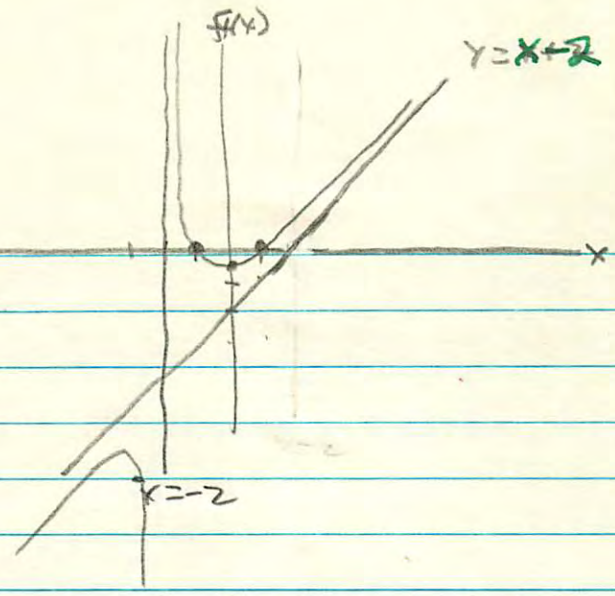
$-2-\sqrt{3} \qquad -2+\sqrt{3}$

Another version

$x(x+4)+1 > 0$ when x large $+(+)+1>0$

when x large neg $-(-)+1>0$

Inbetween $f' < 0$.



a few other things to Notice:

$f(0) = -\frac{1}{2}$

$f(\pm 1) = 0$

$\lim_{x \to -2^+} f(x) = +\infty$

$\lim_{x \to -2^-} f(x) = -\infty$

Not really necessary

$f(-2+\sqrt{3}) = \frac{4+3-6\sqrt{3}+1}{\sqrt{3}} - \frac{8-6\sqrt{3}}{\sqrt{3}} = \frac{8\sqrt{3}-18}{3}$

$f(-2-\sqrt{3}) =$

$f'' = \frac{(2x+4)(x+2)^2 - 2(x-2)(x^2+4x+1)}{(x+2)^4}$

$= \frac{2(x+2)^3 - 2(x-2)(x^2+4x+1)}{(x+2)^4}$

$= \frac{2(x+2)^2 - 2(x^2+4x+1)}{(x+2)^3}$

check at $x = -2+\sqrt{3}$

we get $\frac{2(+) - 2 \cdot 0}{+} = +$

check at $x = -2-\sqrt{3}$

we get $\frac{2(+) - 2 \cdot 0}{-} = -$

∴ by f" test we have min at $-2+\sqrt{3}$ and a max at $-2-\sqrt{3}$

Now we can complete sketch

eg 1 graph $f(x) = x^2 e^{-x}$

① Domain? $\forall x \in \mathbb{R}$ Restrictions here cause vertical asymptotes.

② cross x axis? (when $f(x) = 0$)

$0 = x^2 e^{-x} \Rightarrow x = 0$

③ cross y axis? $f(x) = 0$ $x = 0$

④ horizontal tangents?

$f'(x) = 2xe^{-x} - x^2 e^{-x}$

$e^{-x}(2x - x^2) \overset{\text{set}}{=} 0$

$x(2-x) = 0$

Critical Values of $x \Rightarrow$ $\boxed{x = 0 \quad x = 2}$

⑤

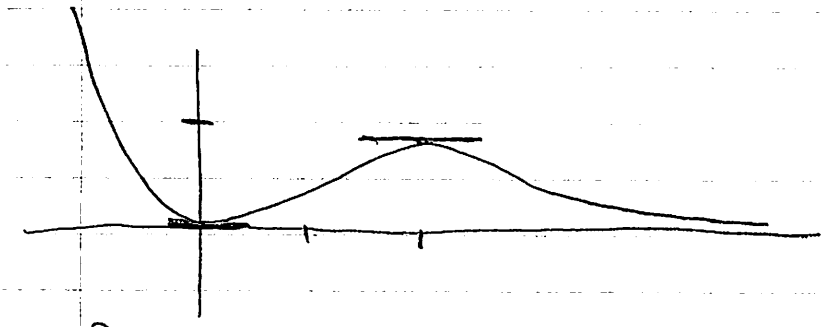
$f(0) = 0$	$f(2) = 4e^{-2}$	coordinates of ⇐ Critical PTS
$(0, 0)$	$(2, \frac{4}{e^2})$	

⑥ check of slant of tangents. f' $(2x - x^2)$

$- \; 0 \; + \; 0 \; -$ (0, 2)

$x(2-x) > 0$ Positive Slope $0 < x < 2$

$x > 0 \quad 2 - x > 0$ $\quad x < 2$ Neg slope $x < 0$ or $x > 2$



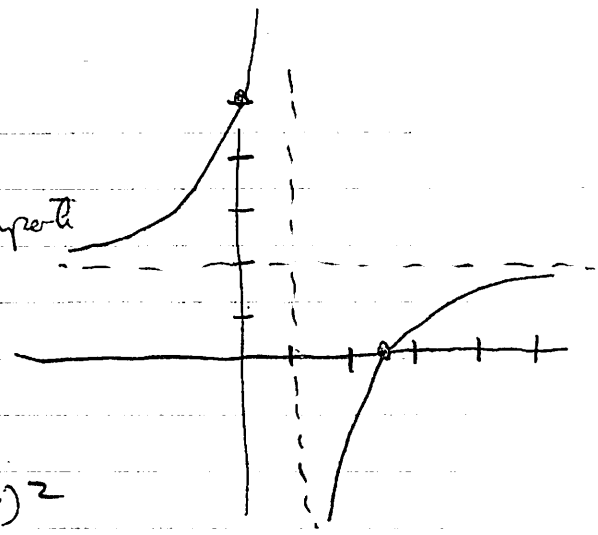
⑦ Behavior $x \to \infty, -\infty$?

$\lim_{x \to \infty} x^2 e^{-x} = \lim_{x \to \infty} \frac{x^2}{e^x} = 0$ horizontal asymptote,

$\lim_{x \to -\infty} x^2 e^{-x} = \lim_{a \to \infty} x^2 e^{a} \Rightarrow +\infty$

we have local max at $(2, \frac{4}{e^2})$ and an absolute min at $(0, 0)$.

eg 2 graph $f(x) = \frac{2x-5}{x-1}$



① Domain? $x \neq 1$, Vert asympt
② x intercept? $(\frac{5}{2}, 0)$
③ y intercept? $(0, 5)$
④ horizontal tangents?

$$f'(x) = \frac{2(x-1) - (2x-5)}{(x-1)^2} = \frac{3}{(x-1)^2}$$

No critical values.

⑤ fctn increasing everywhere since $f'(x) > 0 \ \forall x$

⑥ $\lim_{x\to\infty} \frac{2x-5}{x-1} = \lim_{x\to\infty} \frac{2 - \frac{5}{x}}{1 - \frac{1}{x}} = 2$ horizontal asymptote

$\lim_{x\to 1^+} \frac{2x-5}{x-1} \quad \frac{-}{+} \Rightarrow -\infty$

$\lim_{x\to 1^-} \frac{2x-5}{x-1} \quad \frac{-}{-} \Rightarrow +\infty$

$\lim_{x\to -\infty} \frac{2x-5}{x-1} = 2$

Natural we have No max or min

§3 $f(x) = 2x^3 - 3x^2 - 12x$

① Domain? $\forall x$

② x intercept? $2x^3 - 3x^2 - 12x = 0$

$x(2x^2 - 3x - 12) = 0$

$x = \frac{3 \pm \sqrt{9+96}}{4}$

$x = \frac{3 \pm \sqrt{105}}{4}$ $\frac{+B}{4}$

③ $f(0) = 0$,

④ critical values? $f' = 6x^2 - 6x - 12$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$f'(0) < 0$ So

	2	-3	-12	0
$f(2) =$	2	1	-10	-20
$f(-1) =$	2	-5	-7	7

⑤ + 0 − 0 + $x = 2, -1$

$x-2 > 0$ $x+1 > 0$

$x > 2$ $x > -1$ $\Rightarrow x > 2$

tangent > 0 $x > 2$, $x < -1$

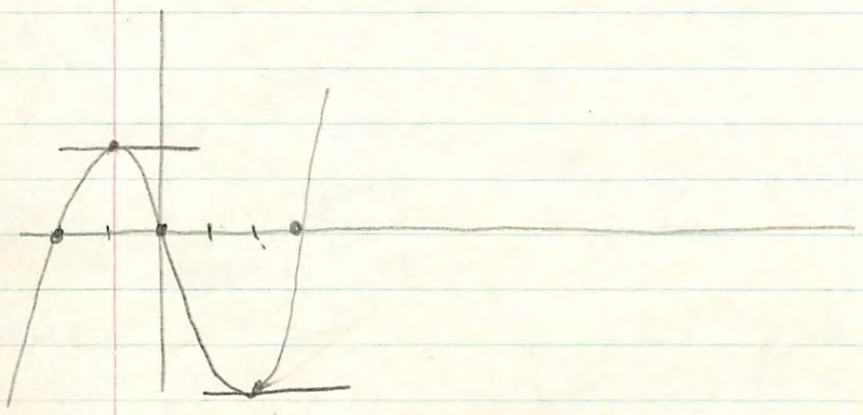
tangent $< -1 < x < 2$

⑥ obviously $\lim_{x \to \infty} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = -\infty$.

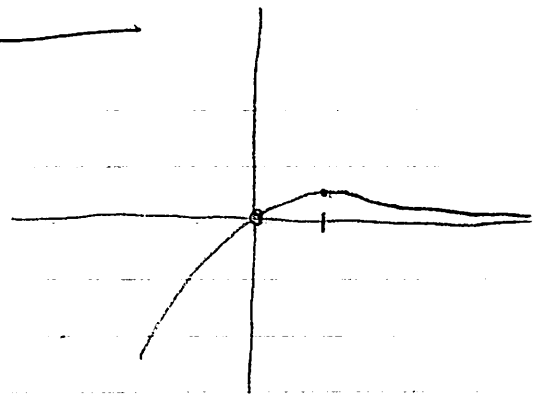
Natural Local max at $(-1, 7)$

Local min at $(2, -20)$

There are No global or absolute max or min.



① $f(x) = xe^{-x}$



$f'(x) = e^{-x} - xe^{-x}$

$e^{-x}(1-x) \overset{\text{set}}{=} 0$

$x = 1$

$f(1) = \frac{1}{e} \approx \frac{1}{3}$

$f'(x) > 0 \quad x < 1$

$f'(x) < 0 \quad x > 1$

$\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = 0$

$\lim_{x\to-\infty} f(x) = -\infty$

② $f(x) = x^2 - 6x + 5$

$(x-5)(x-1)$

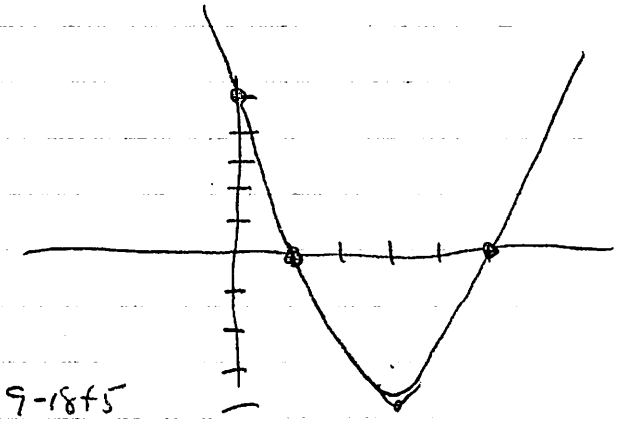
$x = 5, 1$

$f(0) = 5$

$f'(x) = 2x - 6$

$2(x-3) \overset{\text{set}}{=} 0$ $\quad f(3) = 9 - 18 + 5$

$f'(x) > 0, \; x > 3$



④ $f(x) = \frac{x}{x^2+1}$ $\quad D_f \; x \in \mathbb{R}$

$f(0) = 0$

$f'(x) = \frac{x^2+1-2x^2}{(x^2+1)^2} \overset{\text{set}}{=} 0$

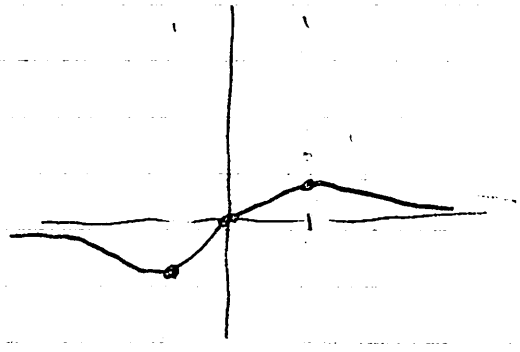
$x = \pm 1$

$f(\pm 1) = \pm \frac{1}{2}$

$\frac{1-x^2}{(x^2+1)^2}$

$f'(x) > 0 \quad |x| < 1$

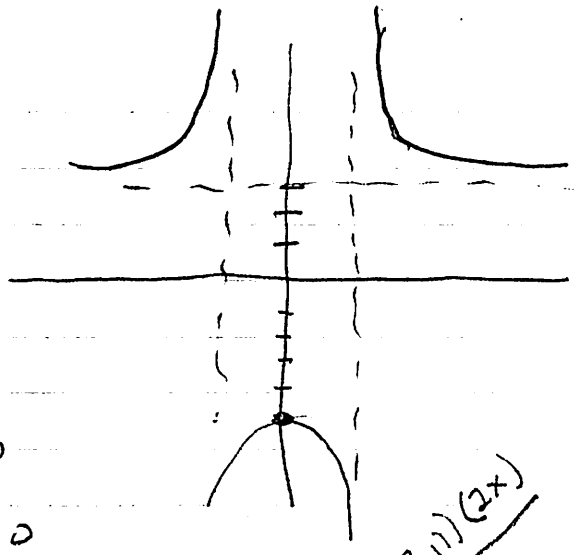
$f'(x) < 0 \quad |x| > 1$



$\lim_{x\to\infty} \frac{x}{x^2+1} = 0$

$\lim_{x\to-\infty} \frac{x}{x^2+1} = 0$

⑤ $f(x) = \dfrac{3x^2+5}{x^2-1}$ $x \neq \pm 1$



$f(0) = -5$ Doesn't cross x axis

$f'(x) = \dfrac{6x(x^2-1) - (6x^3+10x)}{(x^2-1)^2}$

$\dfrac{-16x}{(x^2-1)^2}$ $f' < 0$ $x > 0$

$f' > 0$ $x < 0$

$\lim_{x \to 1^-} \dfrac{3x^2+5}{x^2-1} = \dfrac{+}{-} \Rightarrow -\infty$

$\lim_{x \to -1^+} f(x) = \dfrac{+}{-} \Rightarrow -\infty$

$\lim_{x \to 1^+} f(x) = +\infty$

$\lim_{x \to \infty} f(x) = 3$

$\lim_{x \to -\infty} f(x) = 3$

$\lim_{x \to -1^-} f(x) = \dfrac{+}{+} = +\infty$

$f'' \quad \dfrac{-16(x^2-1)^2 + 16x(2(x^2-1))(2x)}{(\quad)^4}$

$\dfrac{-16(x^2-1)^2 + 64x^2(x^2-1)}{(\quad)^4}$

$\dfrac{-16(x^2-1) + 64x^2}{(\quad)^3}$

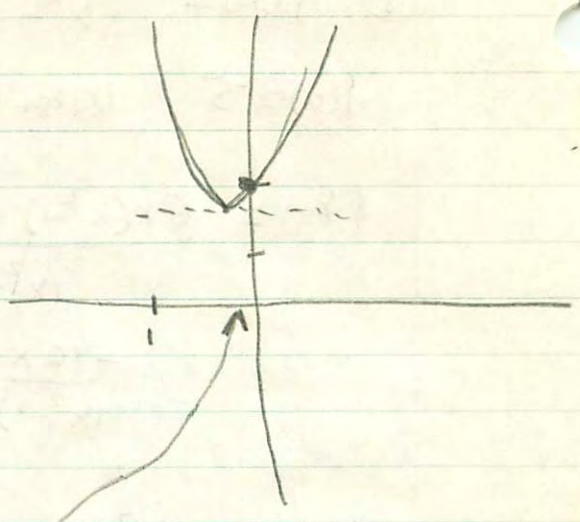
$\dfrac{16(-x^2+1+4x^2)}{(\quad)^3}$

$\dfrac{16(3x^2+1)}{(x^2-1)^3}$

200

$e^{2x} + \frac{1}{e^x} \Rightarrow$

⑥ $e^{2x} + e^{-x} = f(x)$ $\qquad \frac{e^{3x}+1}{e^x} =$



$f(0) = 2$

$f(x) \neq 0$

$f'(x) = 2e^{2x} - e^{-x} \overset{Set}{=} 0$

$2e^{2x} - \frac{1}{e^x}$

$\frac{2e^{3x} - 1}{e^x} \overset{Set}{=} 0$

$e^{3x} = \frac{1}{2}$

$3x = -\ln 2$

$x = -\frac{1}{3}\ln 2 \approx -.23$

$f'(x) > 0 \quad 2e^{3x} > 1$

$f'(x) < 0 \quad 2e^{3x} < 1$

$f(\ln 2^{-\frac{1}{3}}) = e^{2\ln 2^{-\frac{1}{3}}} + e^{-\ln 2^{-\frac{1}{3}}}$

$e^{\ln 2^{-2/3}} + e^{\ln 2^{\frac{1}{3}}}$

$2^{-2/3} + 2^{\frac{1}{3}}$

$\frac{1}{2^{2/3}} + 2^{\frac{1}{3}}$

$\frac{1+2}{2^{2/3}} = \frac{3}{2^{2/3}} \approx 1.9$

$+\infty$

$\lim_{x \to \infty} e^{2x} + e^{-x} \Rightarrow 0 = +\infty$

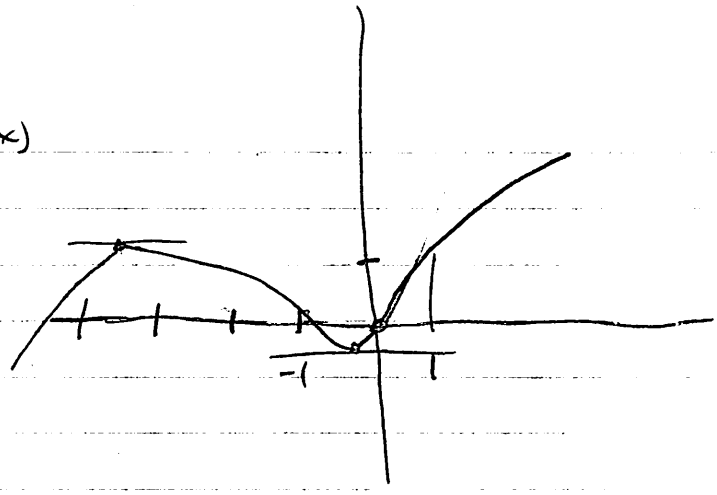
$\lim_{x \to -\infty} f(x) = +\infty.$

$x + 2\ln(1+x^2) = 0$

$\ln(1+x^2) = -\frac{1}{2}x$

$1+x^2 = e^{-\frac{x}{2}}$

(8) $x + 2\ln(1+x^2) = f(x)$



$f(0) = 0$ $\quad D_f \quad x \in \mathbb{R}$

$f'(x) = 1 + \frac{4x}{1+x^2}$

$f'(x) = \frac{x^2+4x+1}{x^2+1} \overset{\text{set}}{=} 0$

$x = \frac{-4 \pm \sqrt{16-4}}{2}$

$x = \frac{-4 \pm 2\sqrt{3}}{2}$

$x = -2 \pm \sqrt{3}$

$f(-2+\sqrt{3}) = -2+\sqrt{3} + 2\ln(1+(-2+\sqrt{3})^2)$

$= -2+\sqrt{3} + 2\ln(8-4\sqrt{3})$

$+ .139$

$= -1.3$

$f(-2-\sqrt{3}) = -2-\sqrt{3} + 2\ln(8+4\sqrt{3})$

$+ 5.41$

$= 1.67$

$(-2+\sqrt{3})^2 = 4+3-4\sqrt{3}$

$7-4\sqrt{3}$

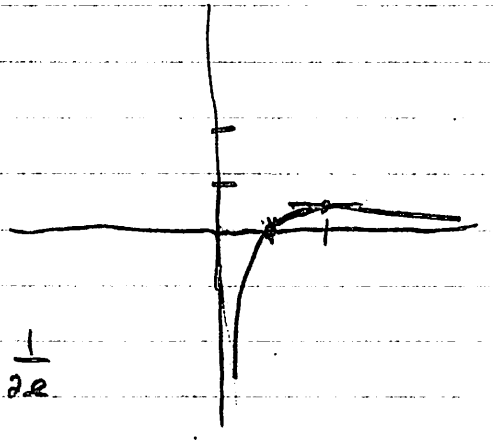
$f'(x) < 0 \quad -2-\sqrt{3} < x < -2+\sqrt{3}$ (checked $-2+0 = x = -2$ $f'(-2)$.)

$\therefore f'(x) > 0 \quad -2-\sqrt{3} > x, \quad x > -2-\sqrt{3}$

too hard to locate 2nd x intercept.

(9) $\frac{\ln x}{x^2} = f(x) \qquad x > 0$

$f(x) = 0, \quad x = 1$



$f'(x) = \frac{x - 2x\ln x}{x^4}$

$1 - 2\ln x = 0$

$\ln x^2 = 1$

$x^2 = e$

$x = e^{\frac{1}{2}}$

$\frac{\frac{1}{2}}{e} = .18 = \frac{1}{2e}$

$\lim_{x \to \infty} \frac{\ln x}{x^2} = 0$

$\lim_{x \to 0^+} \frac{\ln x}{x^2} = \frac{-}{+} \Rightarrow -\infty$

Require students to give coord of critical pts.

25 $\frac{\ln x}{x}$ $[1, 10]$

$f(1) = 0$

$f(e) = \frac{1}{e} \approx .37$

$f(10) = .23$

$f'(x) = \frac{1 - \ln x}{x^2} \overset{\text{set}}{=} 0$

$\ln x = 1$

$x = e$

max $(e, \frac{1}{e})$

min $(1, 0)$

(26) $f(x) = 4x - x^2$ $[0, 5]$

$f(0) = 0$

$f(2) = 4$

$f(5) = -5$

$f'(x) = 4 - 2x$

$x = 2$

min $(+5, -5)$ max $(2, 4)$

(27) $f(x) = 4x - x^2$ $[0, 1]$

$f(0) = 0$ min

No critical pts w/in interval $f(1) = 3$ max

(28) $f(x) = 2x^2 - 5x$ $[-1, 1]$

$f(-1) = 7$ max

$f'(x) = 4x - 5$

$f(1) = -3$ min.

$x = \frac{5}{4}$ reject.

(29) $f(x) = x^3 - 2x^2 + 5x$ $[-1, 3]$

$f'(x) = 3x^2 - 4x + 5$

$x = \frac{4 \pm \sqrt{16 - 60}}{\text{reject}}$ No critical pts

	1	-2	5	0	
-1	1	-3	8	-8	min
3	1	1	8	24	max

(30) $\frac{x}{x^2+1} = f(x)$ $[0, 3]$

$f(0) = 0$ min

$f(1) = \frac{1}{2}$ max.

$f(3) = \frac{3}{10}$

$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)}$

$1 - x^2$

reject $x = -1$

NOT IN interval

(31) $e^{-x}\sin x = f(x) \quad [0, 2\pi]$ (u, v)

$f(0) = 0$ min

$f\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}e^{-\frac{\pi}{4}}}{2}$ max

$f^{-1}(x) = -e^{-x}\sin x + e^{x}\cos x$

$f\left(\frac{7\pi}{4}\right) = \frac{-\sqrt{2}e^{-\frac{5\pi}{4}}}{2}$ min

$\sin x = \cos x$

$x = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$

$f(2\pi) = 0$

(32) $\frac{(x+1)}{\sqrt{x^2+1}} = f(x) \quad [0, 3]$

$f(0) = 1$ min

$f(1) = \frac{2}{\sqrt{2}} = \sqrt{2}$ max 1.4

$f(3) = \frac{4}{\sqrt{10}} = 1.2$

$v = \sqrt{x^2+1}$

$du = \frac{x}{\sqrt{x^2+1}}$

$$\frac{\sqrt{x^2+1} - \frac{x(x+1)}{\sqrt{x^2+1}}}{(x^2+1)^2}$$

$$\frac{x^2+1-x^2-x}{(x^2+1)^{5/2}}$$

$x = 1$

2nd Derivative

a curve is said to be concave upward if for (a,b) $f'' > 0$

if $f'' < 0$, (a,b) F is concave down,

2nd derivative test if f'' test, for max/min.
for $x = c$ a critical value of f(x)
$f''(c) > 0$ min concave up

$f''(c) < 0$ max concave down.

Notice $f''(c) = 0$ is Not defined.

graph (eg 2 pg 211) $f(x) = xe^x$ $f(-1) = \frac{-1}{e}$

$f'(x) = e^x + xe^x \overset{\text{set}}{=} 0$

$e^x(1+x) = 0 \Rightarrow x = -1 = c$

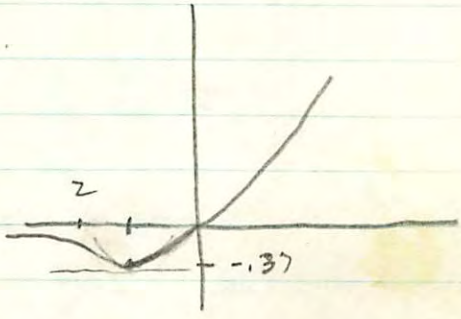
$f''(x) = 2e^x + xe^x = e^x(2+x)$ $f'' < 0$ $x < -2$
$f'' > 0$ $x > -2$

$f''(-1) \geq 0$ ∴ min at $(-1, f(-1))$ concave up.

$f(x) = 0$ $x = 0$ only if one I crossing.

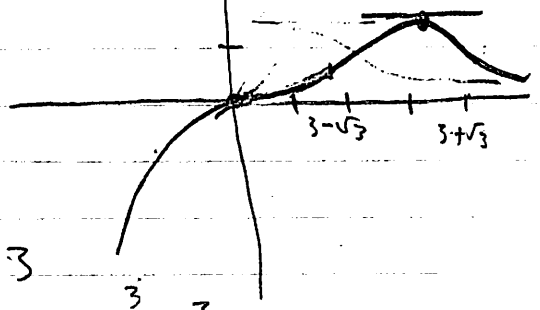
$\lim_{x \to -\infty} xe^x = \lim_{a \to +\infty} \frac{a}{e^a} = 0$

$\lim_{x \to +\infty} f(x) = \infty$



Lecture

eg 4 (pg 213) $f(x) = x^3 e^{-x}$



$0 = x^3 e^{-x} \Rightarrow x = 0$ 1 crossing

$f'(x) = 3x^2 e^{-x} - x^3 e^{-x} \overset{\text{set}}{=} 0$

$x^2 e^{-x}(3-x) = 0$ 0, 3

$f(3) = 3^3 e^{-3} = 1.3$

$f''(x) = 6x e^{-x} - 3x^2 e^{-x} - [3x^2 e^{-x} - x^3 e^{-x}]$

$= 6x e^{-x} - 6x^2 e^{-x} + x^3 e^{-x}$

$f''(x) = x e^{-x}(6 - 6x + x^2) \iff$ does ~ $\frac{6 \pm \sqrt{36-24}}{2}$

$\frac{6 \pm \sqrt{12}}{2} = 3 \pm \sqrt{3}$

$f''(0) = 0$ No help

$f''(3) = 3(e^{-3})(6 - 18 + 9) = -9e^{-3} < 0$ ∴ max

$\lim_{x \to \infty} x^3 e^{-x} = 0$

$f''(x) = x e^{-x}(6 - 6x + x^2) \geq 0$

$x \in \{0, 3 \pm \sqrt{3}\}$

$- \; 0 \; + \; 0 \; - \; 0 \; +$

0 $3-\sqrt{3}$ $3+\sqrt{3}$

$f(100) > 0$

W.T.P. $\sqrt{ab} < \frac{1}{2}(a+b)$ $\quad a < c < b$

MVT guarantees $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Let $f(x) = \sqrt{x}$

$f'(x) = \dfrac{1}{2\sqrt{x}} \rightarrow \dfrac{1}{2\sqrt{c}} = \dfrac{f(b)-f(a)}{b-a}$

$a < c < b$

$a > 0$

so

$\sqrt{a} < c < b$

$\dfrac{1}{2\sqrt{c}} = \dfrac{\sqrt{b}-\sqrt{a}}{b-a}$

$\dfrac{b-a}{2\sqrt{c}} = \sqrt{b}-\sqrt{a}$

$\dfrac{1}{\sqrt{b}} < \dfrac{1}{\sqrt{c}} < \dfrac{1}{\sqrt{a}}$ *

$\dfrac{b-a}{2\sqrt{c}} = \sqrt{b}-\sqrt{a} < \dfrac{b-a}{2\sqrt{a}}$ from **

$\dfrac{b-a}{\sqrt{a}} > \dfrac{b-a}{\sqrt{c}} > \dfrac{b-a}{\sqrt{b}}$

$\sqrt{ab} - \sqrt{a^2} < \frac{1}{2}b - \frac{1}{2}a$

$\dfrac{b-a}{2\sqrt{a}} > \dfrac{b-a}{2\sqrt{c}} > \dfrac{b-a}{2\sqrt{b}}$ **

$\sqrt{ab} - a <$

$\sqrt{ab} < \frac{1}{2}b + \frac{1}{2}a$

$\sqrt{ab} < \dfrac{b+a}{2}$

* $\sqrt{a} < \sqrt{c}$ so dividing by smaller number yields larger quotient.

Using the Mean Value Thm

Prove $\sqrt{ab} < \frac{1}{2}(a+b)$

$f(x) = \sqrt{ab} - \frac{1}{2}(a+b)$

$a < c < b$

MVT: $\frac{f(b)-f(a)}{b-a} = f'(c)$

Let $f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\frac{f(b)-f(a)}{b-a} = \frac{1}{2\sqrt{c}}$

$\sqrt{b}+\sqrt{a} = \frac{1}{2\sqrt{c}}$

$\sqrt{b}+\sqrt{a} > \frac{1}{2\sqrt{a}}$

$2\sqrt{ab}+2a > 1$

$\frac{\sqrt{b}-\sqrt{a}}{b-a} = \frac{1}{2\sqrt{c}}$

$a < c < b$

$\sqrt{a} < \sqrt{c} < \sqrt{b}$

$\sqrt{b}-\sqrt{a} = \frac{b-a}{2\sqrt{c}}$

$\frac{1}{\sqrt{a}} > \frac{1}{\sqrt{c}} > \frac{1}{\sqrt{b}}$

$\sqrt{b}-\sqrt{a} < \frac{b-a}{2\sqrt{a}}$

$\frac{b-a}{2\sqrt{a}} > \frac{b-a}{2\sqrt{c}} > \frac{b-a}{2\sqrt{b}}$

$\sqrt{ab} - a < \frac{1}{2}b - \frac{1}{2}a$

$\sqrt{ab} < \frac{1}{2}b + \frac{1}{2}a$

$\sqrt{ab} < \frac{1}{2}(a+b)$

$a < c$ so dividing by a smaller number yields a larger value

$$\frac{1}{\pi^2} = 4x^4 - 4x^2$$

$$4\pi^2 x^4 - 4\pi^2 x^2 - 1 = 0$$

$$(2\pi x^2 \quad 1)(2\pi x^2 \quad 1)$$

6.7
Pg 215

① $f(x) = x^3 - 6x^2 - 15x$

$x(x^2 - 6x - 15)$

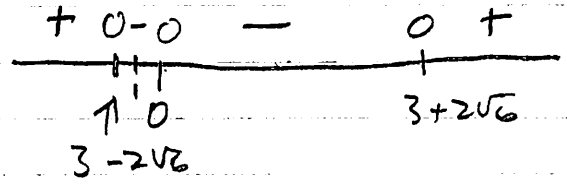
$x \quad 3 \pm 2\sqrt{6}$

$\frac{6 \pm \sqrt{36+60}}{2}$

$\frac{6 \pm \sqrt{96}}{2}$

$\frac{6 \pm 4\sqrt{6}}{2}$

16
6

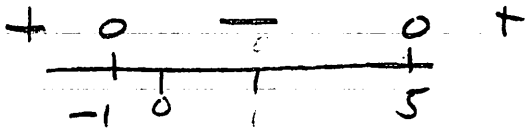


$f(1) = 1 - 6 - 15 < 0$

$f'(x) = 3x^2 - 12x - 15$

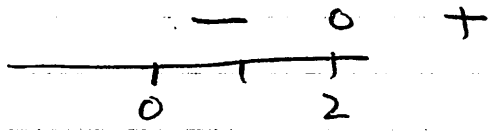
$3(x^2 - 4x - 5)$

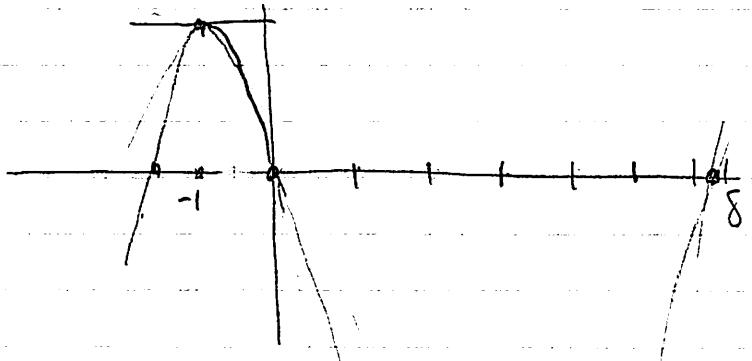
$3(x-5)(x+1)$



$f''(x) = 6x - 12$

$6(x-2)$





$f = 1 \quad -6 \quad -15 \quad 0$

$-1 \mid 1 \quad -7 \quad -8 \quad 8$

$2 \mid 1 \quad -4 \quad -25 \quad -50$

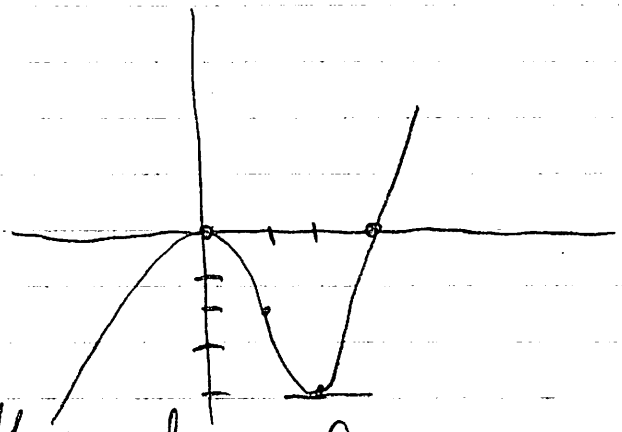
$5 \mid 1 \quad -1 \quad -20 \quad -100$

② $y = x^3 - 3x^2$ $\quad x^2(x-3)$

$y' = 3x^2 - 6x$ $\quad 3x(x-2)$

$y'' = 6x - 6$ $\quad 6(x-1)$

	1	-3	0	0
2	1	-1	-2	-4
1	1	-2	-2	-2



OK window 1980 wd.

③ $y = \frac{x^2}{2} + \frac{1}{x}$

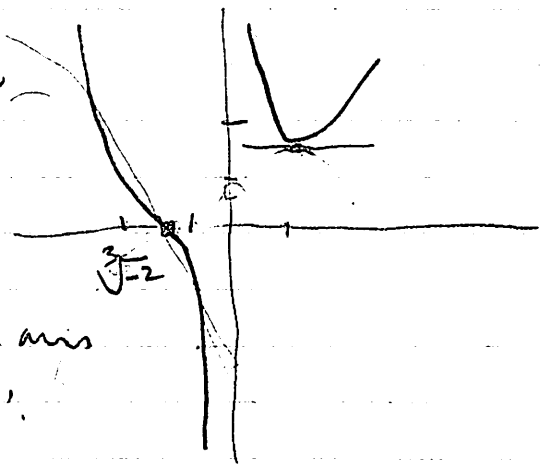
$y' = x - \frac{1}{x^2}$

$y'' = 1 + \frac{2}{x^3}$

$y'(-2)$ $\quad y'(-1)$ $\quad$ LOPI

$f''(1) > 0$ ∴ min

$f''(-1) < 0$ ∴ $f''(-2) < 0$



$\sqrt[3]{-2}$

$1 = \frac{x^2}{2} + \frac{1}{x}$

$2x = x^3 + 2$

$x^3 - 2x + 2 = 0$

∴ this has a solution.

$f(0) = 2$

$f(-1) = -1$

$\frac{x^2}{2} + \frac{1}{x} = 0$ $\quad x^2 - 1 = 0$

$x^3 = -2$

$x = \sqrt[3]{-2}$

$\lim_{x \to \infty} \frac{x^2}{2} + \frac{1}{x} = \infty$

f''

$x^3 = -2$

$x^3 + 2 = 0$

$f'' > 0$ $x^3 > -2$

$f'' <$ $x^3 < -2$

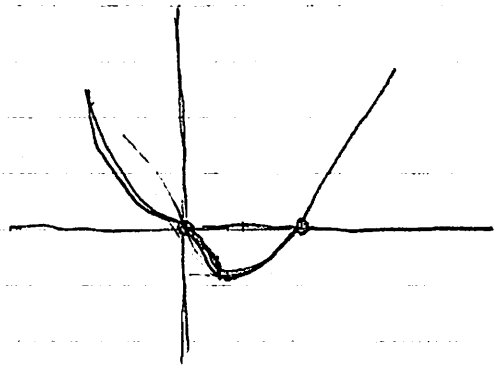
④ $f(x) = 2x^4 - 4x^3 = 2x^3(x-2)$

$f'(x) = 8x^3 - 12x^2 = 4x^2(2x-3)$

$f''(x) = 24x^2 - 24x = 24x(x-1)$

$24x > 0$ & $x-1 > 0$

$x > 0$ $\quad x > 1$



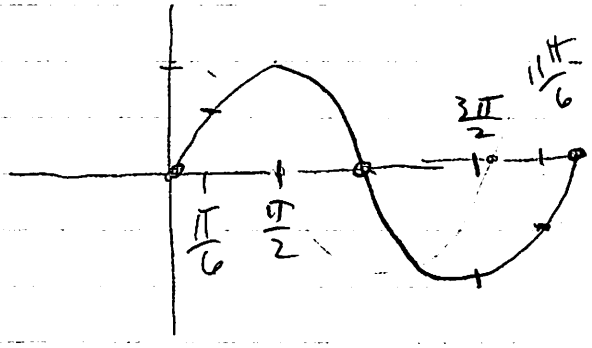
$f \to \quad 2 \quad -4 \quad 0 \quad 0$

$\frac{2}{3} \quad 2 \quad -\frac{8}{3} \quad -\frac{16}{9} \quad -\frac{32}{27} \quad -\frac{64}{81}$

⑥ $f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$



⑦ $y = e^{-x^2}$

$y' = -2xe^{-x^2}$

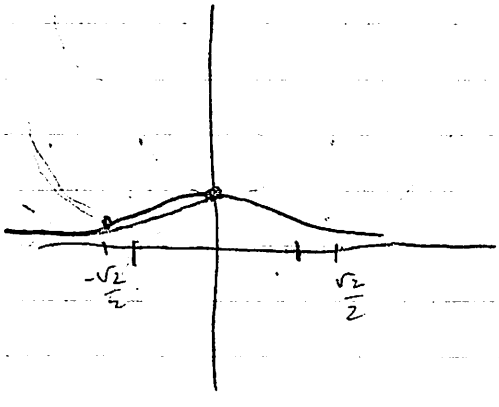
$y'' = -2e^{-x^2} + 4x^2e^{-x^2}$

$e^{-x^2}(-2+4x^2)$

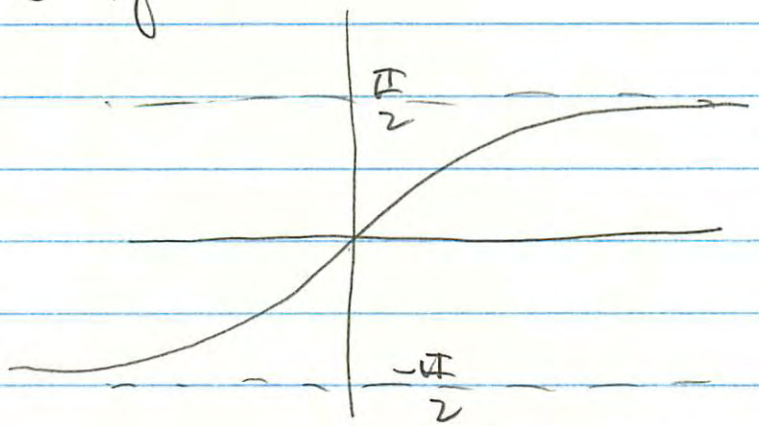
$4x^2 - 2 > 0$

$x^2 - \frac{1}{2} > 0$

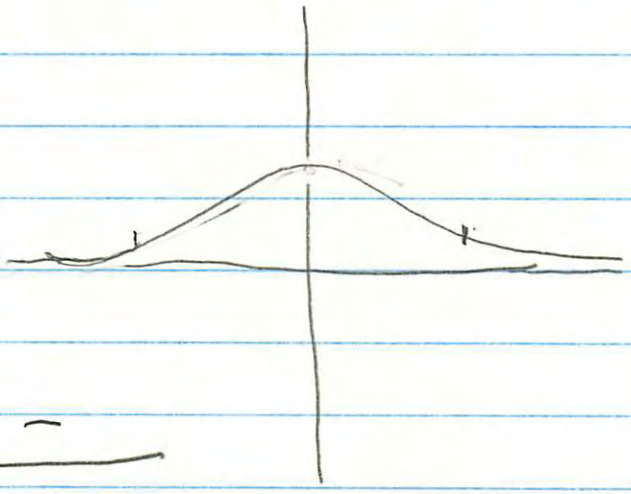
$x^2 = \frac{1}{2}$



#18 Pg 215



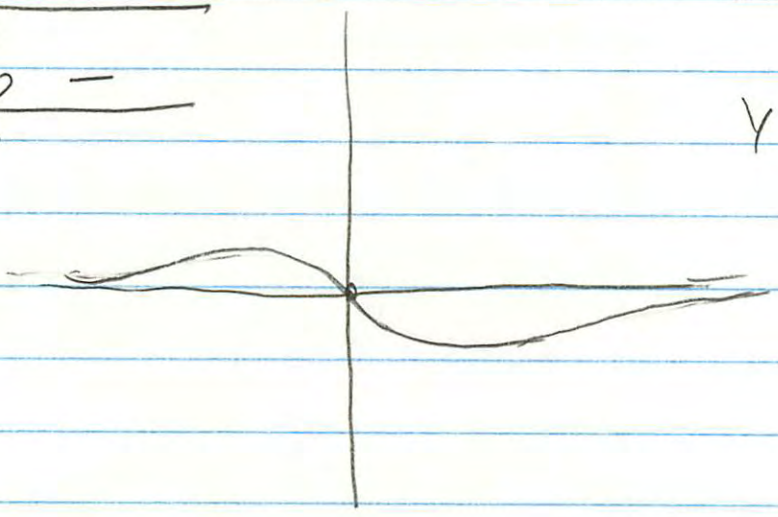
$y = \tan^{-1} x$



$\frac{1}{x^2+1}$

$y' = \frac{1}{x^2+1}$

y	+	0	−
y'	+		+
y''	+	0	−

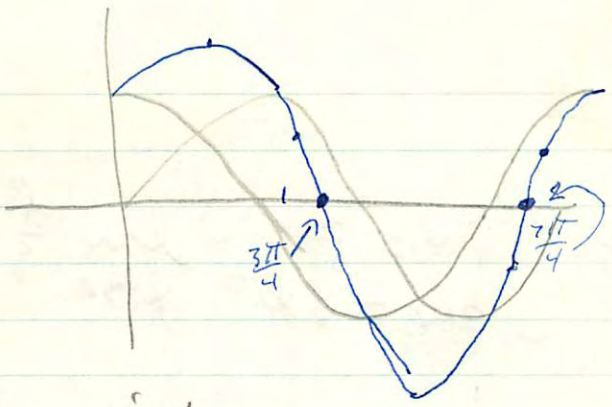


$y'' = \frac{-2x}{(x^2+1)^2}$

(15) $y = \sin x + \cos x$

$y' = \cos x - \sin x$

$y'' = -\sin x - \cos x$



(18) $y = \tan^{-1} x$

$y' = \frac{1}{x^2+1}$

$y'' = \frac{-2x}{(x^2+1)^2}$

See previous page

(19) $y = x^2 e^{-x}$

$y' = -x^2 e^{-x} + 2x e^{-x}$

$y'' = x^2 e^{-x} - 2x e^{-x} + 2e^{-x} - 2x e^{-x}$

$\lim_{x\to 0} f = 0$ $f(x) = 2x$ $\lim_{x\to 0} \frac{f}{g} = \frac{2}{3}$

$\lim_{x\to 0} g = 0$ $g(x) = 3x$

$f(x) = 1 - \cos x$

$g(x) = x$

$\lim_{x\to 0} \frac{f}{g} = 0$

$\lim_{x\to 0} f = 0$ $f(x) = x^2$

$g(x) = 3x$

$\lim_{x\to 0} g = 0$

$f(x) = x$ $\lim_{x\to 0} \frac{f}{g} = \infty$

$g(x) = x^2$

$\lim_{x\to 0} f = 0$

$\lim_{x\to 0} g = 0$

$f = \sin x$ $\lim_{x\to 0} \frac{f}{g} = 1$

$g = x$

$\lim_{x\to 0} f = 0$

$\lim_{x\to 0} g = 0$

Begin:

Take out calculators.

Zoom STANDARD.

$Y_1 = \dfrac{x^5 - 1}{x^3 - 1}$

graph

zoom/box around $x = 1$

Trace when $x = 1$ $y = \frac{5}{3}$.

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1}$$

$$\lim_{x \to 1} \frac{(x-1)(x^4 + x^3 + x^2 + x + 1)}{(x-1)(x^2 + x + 1)}$$

$$\frac{5}{3}$$

$Y_2 = (6(\sin x - x)) / x^3$

zoom std.

zoom/box around $x = 0$

$x = 0$, $y = -1$

$$\lim_{x \to 0} \frac{6 \sin x - x}{x^3}$$

No current techniques have much effect.

introduce l'Hopital

$$\lim_{x \to 1} \frac{x^5 - 1}{x^3 - 1} \quad \frac{0}{0}$$

$$\lim_{x \to 1} \frac{5x^4}{3x^2}$$

$$\frac{5}{3}$$

$$\lim_{x \to 0} \frac{6(\sin x - x)}{x^3} \quad \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\not{6}^{2}(\cos x - 1)}{\not{3} x^2} \quad \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\not{2}(-\sin x)}{\not{2} x} \quad \frac{0}{0}$$

$$\lim_{x \to 1} \frac{-\cos x}{1}$$

$$-1$$

No pick up missing items from old notes

l'hopital's Rule.

6.10

pg 232-240

given $\lim_{x\to a} f(x) = 2$, $\lim_{x\to a} g(x) = 3$

find $\lim_{x\to a} \frac{f(x)}{g(x)}$; $\frac{2}{3}$.

given $\lim_{x\to a} f(x) = 0$ $\lim_{x\to a} g(x) = 0$

find $\lim_{x\to a} \frac{f(x)}{g(x)}$ eg.

$\lim_{x\to 0} \frac{\sin x}{x} = 1$ $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

So no conclusions can be drawn from the $\frac{0}{0}$ situation.

l'hopital's Rule, zero over zero case.
Let a be a number & let g & f be differentiable over some interval (a, b). Assume $g'(x)$ is NOT zero for any x in that interval.

if $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$

and if $\lim_{x\to a} \frac{f'(x)}{g'(x)} = L$ then

$\lim_{x\to a} \frac{f(x)}{g(x)}$ also $= L$

eg. $\lim_{x\to 1^+} \frac{x^5-1}{x^3-1}$ => Notice $f(1) = g(1) = 0$. If the limit of the quotient of the derivatives exists it will be the same as the original limit.

$$\lim_{x\to 1^+} \frac{x^5-1}{x^3-1} = \lim_{x\to 1^+} \frac{5x^4}{3x^2} = \frac{5}{3}.$$

In order to Justify l'Hopital's Rule:

Since $\lim_{x\to a} f(x)=0$ & $\lim_{x\to a} g(x)=0$ it follows that $f(a)=0$ & $g(a)=0$ because f, g are continuous

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)}$$ Both f(a) & g(a) are zero.

$$= \lim_{x\to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$ multiply by $\frac{\frac{1}{x-a}}{\frac{1}{x-a}}$

$$= \frac{\text{Def of Der.}}{\frac{f'(a)}{g'(a)}}$$ Both at the a Because we want f'(x) not f'(a)

$$= \frac{\lim_{x\to a} f'(x)}{\lim_{x\to a} g'(x)}$$

If lim equals L then it still equals L.

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = L.$$

Note l'Hopital's rule applies not only to a specific # such as a ($x\to a$) but $x\to a^+$ or $x\to a^-$ as well as $x\to\infty$ or $x\to-\infty$.

Lecture

eg 2.

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{\cos x - 1}{3x^2} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{-\sin x}{6x} \overset{\frac{0}{0}}{=} \lim_{x\to 0} \frac{-\cos x}{6} = -\frac{1}{6}.$$

Don't need to use the Lopital argument this time, but will use that argument more

Consider $f(x) = x \qquad g(x) = e^x$

$$\lim_{x\to\infty} f(x) = \infty \qquad \lim_{x\to\infty} g(x) = \infty$$

$$\lim_{x\to\infty} \frac{x}{e^x} = ? \quad \frac{\infty}{\infty} \text{ sowegot } \lim_{x\to\infty} \frac{1}{e^x} = 0$$

We have done these pblm like this

Consider $\lim_{x\to\infty} \frac{4x+1}{2x} \quad \frac{\infty}{\infty} \quad \lim_{x\to\infty} \frac{4+\frac{1}{x}}{2} = 2$

but $\lim_{x\to\infty} 4x+1 = \infty$ & $\lim_{x\to\infty} 2x = \infty$.

So we change $\lim_{x\to\infty} \frac{4}{2} = 2$.

L'hopital's rule infinity over infinity case.

Let f, g be differentiable & defined $\forall\ x > \underline{X}$ then if

$$\lim_{x\to\infty} f(x) = \infty \text{ & } \lim_{x\to\infty} g(x) = \infty \text{ and if}$$

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \text{ then } \lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

so $\lim_{x\to\infty} \frac{x}{e^x} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$

Turn to pg 239 "As a rule of thumb ..."

$$\lim_{x \to 0} \left(e^{-\frac{1}{x}}\right)^x \qquad \frac{1}{e^{\frac{1}{x}}}$$

$$(0)^0$$

$$\lim_{x \to 0} \left(e^{-\frac{1}{x}}\right)^x \qquad 0^0 \neq 1$$

$$= \frac{1}{e}.$$

$$\lim_{x \to 0} \left(e^{-\frac{1}{x}}\right)^x = \lim_{x \to 0} \left(\frac{1}{e^{\frac{1}{x}}}\right)^x \Rightarrow 0^0$$

$$\frac{1}{e^{\infty}} \Rightarrow 0$$

but we know $\left(e^{-\frac{1}{x}}\right)^x = e^{-1} = \frac{1}{e}$

Conclusion $0^0 = \frac{1}{e}$

Newsgroups: sci.math,sci.answers,news.answers
Path:
odin.community.net!news.zeitgeist.net!ico.net!pacbell.com!decwrl!spool.mu.edu!torn!watserv3.uwaterloo.ca!undergrad.math.uwaterloo.ca!neumann.uwaterloo.ca!alopez-o
From: alopez-o@neumann.uwaterloo.ca (Alex Lopez-Ortiz)
Subject: sci.math FAQ: What is 0^0?
Summary: Part 15 of many, New version,
Originator: alopez-o@neumann.uwaterloo.ca
Message-ID: <DI76Ky.54t@undergrad.math.uwaterloo.ca>
Sender: news@undergrad.math.uwaterloo.ca (news spool owner)
Approved: news-answers-request@MIT.Edu
Date: Fri, 17 Nov 1995 17:14:58 GMT
Expires: Fri, 8 Dec 1995 09:55:55 GMT
Reply-To: alopez-o@neumann.uwaterloo.ca
Nntp-Posting-Host: neumann.uwaterloo.ca
Organization: University of Waterloo
Followup-To: sci.math
Lines: 165
Xref: odin.community.net sci.math:108425 sci.answers:2917 news.answers:48102

Archive-Name: sci-math-faq/specialnumbers/0to0
Last-modified: April 26, 1995
Version: 6.2

What is 0^0

According to some Calculus textbooks, 0^0 is an ``indeterminate form''. When evaluating a limit of the form 0^0 , then you need to know that limits of that form are called ``indeterminate forms'', and that you need to use a special technique such as L'Hopital's rule to evaluate them. Otherwise, 0^0 = 1 seems to be the most useful choice for 0^0 . This convention allows us to extend definitions in different areas of mathematics that otherwise would require treating 0 as a special case. Notice that 0^0 is a discontinuity of the function x^y .

This means that depending on the context where 0^0 occurs, you might wish to substitute it with 1, indeterminate or undefined/nonexistent.

Some people feel that giving a value to a function with an essential discontinuity at a point, such as x^y at (0,0) , is an inelegant patch and should not be done. Others point out correctly that in mathematics, usefulness and consistency are very important, and that under these parameters 0^0 = 1 is the natural choice.

The following is a list of reasons why 0^0 should be 1.

Rotando &Korn show that if f and g are real functions that vanish at the origin and are analytic at 0 (infinitely differentiable is not sufficient), then f(x)^g(x) approaches 1 as x approaches 0 from the right.

From Concrete Mathematics p.162 (R. Graham, D. Knuth, O. Patashnik):

> Some textbooks leave the quantity 0^0 undefined, because the functions x^0 and 0^x have different limiting values when x decreases to 0. But this is a mistake. We must define x^0=1 for all x , if the binomial theorem is to be valid when x = 0 , y = 0 , and/or x = -y . The theorem is too important to be arbitrarily

restricted! By contrast, the function 0^x is quite unimportant.

Published by Addison-Wesley, 2nd printing Dec, 1988.

As a rule of thumb, one can say that 0^0 = 1 , but 0.0^(0.0) is undefined, meaning that when approaching from a different direction there is no clearly predetermined value to assign to 0.0^(0.0) ; but Kahan has argued that 0.0^(0.0) should be 1, because if f(x), g(x) --> 0 as x approaches some limit, and f(x) and g(x) are analytic functions, then f(x)^g(x) --> 1 .

The discussion on 0^0 is very old, Euler argues for 0^0 = 1 since a^0 = 1 for a != 0 . The controversy raged throughout the nineteenth century, but was mainly conducted in the pages of the lesser journals: Grunert's Archiv and Schlomilch's Zeitshrift. Consensus has recently been built around setting the value of 0^0 = 1 .

On a discussion of the use of the function 0^(0^x) by an Italian mathematician named Guglielmo Libri.

> [T]he paper [33] did produce several ripples in mathematical waters when it originally appeared, because it stirred up a controversy about whether 0^0 is defined. Most mathematicians agreed that 0^0 = 1 , but Cauchy [5, page 70] had listed 0^0 together with other expressions like 0/0 and oo - oo in a table of undefined forms. Libri's justification for the equation 0^0 = 1 was far from convincing, and a commentator who signed his name simply ``S'' rose to the attack [45]. August Mvbius [36] defended Libri, by presenting his former professor's reason for believing that 0^0 = 1 (basically a proof that lim_(x --> 0+) x^x = 1). Mvbius also went further and presented a supposed proof that lim_(x --> 0+) f(x)^(g(x)) whenever lim_(x --> 0+) f(x) = lim_(x --> 0+) g(x) = 0 . Of course ``S'' then asked [3] whether Mvbius knew about functions such as f(x) = e^(-1/x) and g(x) = x . (And paper [36] was quietly omitted from the historical record when the collected words of Mvbius were ultimately published.) The debate stopped there, apparently with the conclusion that 0^0 should be undefined.
>
> But no, no, ten thousand times no! Anybody who wants the binomial theorem (x + y)^n = sum_(k = 0)^n (n\choose k) x^k y^(n - k) to hold for at least one nonnegative integer n must believe that 0^0 = 1 , for we can plug in x = 0 and y = 1 to get 1 on the left and 0^0 on the right.
>
> The number of mappings from the empty set to the empty set is 0^0 . It has to be 1.
>
> On the other hand, Cauchy had good reason to consider 0^0 as an undefined limiting form, in the sense that the limiting value of f(x)^(g(x)) is not known a priori when f(x) and g(x) approach 0 independently. In this much stronger sense, the value of 0^0 is less defined than, say, the value of 0 + 0 . Both Cauchy and Libri were right, but Libri and his defenders did not understand why truth was on their side.
>
> [3] Anonymous and S ... Bemerkungen zu den Aufsatze |berschrieben, `Beweis der Gleichung ... , nach J. F. Pfaff', im zweiten Hefte dieses Bandes, S. 134, Journal f|r die reine und angewandte Mathematik, 12 (1834), 292-294.

p 237
lecture.

eg 4 $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ both diverge

1st notice $\lim_{x\to 0^+} \ln x = -\infty$ $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

$$\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -x = 0.$$

eg 5 $\lim_{x\to 0^+} x \ln x$. Since there are no quotients it appears that it does not need the l'Hopital's Rules. So change the Problem so it does fit l'Hopital's rules.

Before eg 6

eg 6 $\lim_{x\to 0^+} x^x = ?$ Let $y = x^x$

$\ln y = x \ln x$.

$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x \ln x$

$= \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{\infty}$

So $\lim_{x\to 0^+} \ln y = 0$ Now we can apply l'Hopital's Rule.

$\lim_{x\to 0^+} y = 1$

$\lim_{x\to 0^+} x^x = 1$.

Lecture.

eg 7 $\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\sin x}\right]$ $\infty-\infty$ Not a l'hop. tal situation, must convert.

$\lim_{x\to 0}\frac{\sin x - x}{x\sin x}$ $\frac{0}{0}$ Now $f=\sin x - x$, $g = x\sin x$

Now it is l'hopital situation.

$\lim_{x\to 0} f = \lim_{x\to 0} g = 0$

$\lim_{x\to 0}\frac{\sin x - x}{x\sin x} = \lim_{x\to 0}\frac{\cos x - 1}{x\cos x + \sin x} =$

$\lim_{x\to 0}\frac{-\sin x}{-x\sin x + 2\cos x} = \frac{-0}{-0+2} = 0$ Thus

$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{\sin x}\right] = 0.$

eg 8 $\lim_{x\to -\infty} e^x x^3$ $\lim_{x\to -\infty} e^x = 0$

$\lim_{x\to -\infty} x^3 = -\infty.$

l'hopital's rule does not hold.

however $\lim_{x\to -\infty}\frac{x^3}{e^{-x}}$ $\lim_{x\to -\infty} e^{-x} = \infty$ Now $\frac{\infty}{\infty}$

$\lim_{x\to -\infty}\frac{x^3}{e^{-x}} = \lim_{x\to -\infty}\frac{3x^2}{-e^{-x}} = \lim_{x\to -\infty}\frac{6x}{e^{-x}} = \lim_{x\to -\infty}\frac{6}{-e^{-x}} = 0$

See first paragraph Pg 239 ... applies when you need it ...

generally l'hopital's Rules can be used when you have: $\frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{\infty}$ $0\cdot\infty$, 0^0, ∞^0 etc.

many times you will have to change the form of the original limit then apply l'hopital's Rules.

$$\lim_{x \to 0} \left(2^{-\frac{1}{x}}\right)^{x} = \lim 2^{-1}$$

$$\lim_{x \to 0} \left(\frac{1}{2^{\frac{1}{x}}}\right)^{x} =$$

$$\left(\frac{1}{2^{\text{Large}}}\right)^{\text{zero}} =$$

$$(\text{zero})^{\text{zero}} = \frac{1}{2}$$

Given $f(x)$, $g(x)$ continuous fcts

$f(a) = 0 \quad g(a) = 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)}$$

$$\lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} \qquad \frac{\frac{1}{x-a}}{\frac{1}{x-a}}$$

$$\lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}}$$

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$\lim_{x \to a} \frac{g(x) - g(a)}{x - a}$$

$$\frac{f'(a)}{g'(a)}$$

$$\lim_{x \to a} \frac{f'(x)}{g'(x)}$$

If this equals L then this also equals L.

$\lim_{x\to 0} \frac{\sin x}{x}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{\cos x}{1} = 1$

$\lim_{x\to 0} \frac{1-\cos x}{x}$ $\frac{1-1}{0}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{\sin x}{1} = 0$

l'hopital's Rule applies for $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$ etc.

See pg 239 "As a rule of Thumb..."

Is this l'hopital situation?

a) $\lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}}$ $\frac{-\infty}{\infty}$ Y.S.

$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$

b) $\lim_{x\to 0} x \ln x$? $0\cdot(-\infty)$ Not l'hopital situations

However ....

See example A above.

$0^0 = ?$

$\lim_{x \to 0} x^x = ?$

$\lim_{x \to 0} \left(2^{-\frac{1}{x}}\right)^x$

$\lim_{x \to 0} \left(\frac{1}{2^{\frac{1}{x}}}\right)^x$

0^0

but $\left(2^{-\frac{1}{x}}\right)^x =$

$2^{-1} = \frac{1}{2}$

so $0^0 = \frac{1}{2}$??

OR let $y = x^x$

$\ln y = x \ln x$

$= \frac{\ln x}{\frac{1}{x}}$

$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \quad \frac{-\infty}{\infty}$

$\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$

$\lim_{x \to 0^+} -x = 0$ so $\lim_{x \to 0} \ln y = 0$ then $e^0 = y$

so $y = 1$

$\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{\sin x} \right]$ $\infty - \infty$ NOT l'hopital situation

$\lim_{x \to 0} \left[\frac{\sin x - x}{x \sin x} \right]$ $\frac{0-0}{0 \cdot 0} = \frac{0}{0}$ IS l'hopital situation

$\lim_{x \to 0} \left[\frac{\cos x - 1}{\sin x + x \cos x} \right]$ $\frac{1-1}{0+0 \cdot 1} = \frac{0}{0}$ STILL l'hopital situation.

$\lim_{x \to 0} \left[\frac{-\sin x}{\cos x + \cos x - x \sin x} \right]$

$\frac{-0}{2-0} = 0$

Goto

http://www.dougshaw.com/findTheERROR/FTELHopitAL.html

↑ CAPITAL Letters.

FIND THE ERROR

It is a beautiful Spring day. You are again jolly, for you have a backpack filled with schoolbooks and cornbread, and you are ready to have your lunch and start working on your calculus homework. You lay out your food and notebook on a picnic table, and suddenly a familiar scent catches your attention. "L'Hopitals rule, eh?" comments a wild eyed stranger who has sat next to you. "Interesting stuff... if you don't mind reading LIES!"

"But L'Hopitals rule is useful! It is a true and wonderful technique of computing limits," you say.

"Useless!" he bellows. "You show me a steak and baked potato with a side of L'Hopital's rule and I'll show you a steak and baked potato, by cracky!"

"What are you talking about?"

"Look at this, and see the lies exposed!"

The stranger writes this in your notebook:

$$\lim_{x\to\infty} \frac{x^2+\sin x}{x^2} \quad \left(\frac{\infty}{\infty}\right) =$$

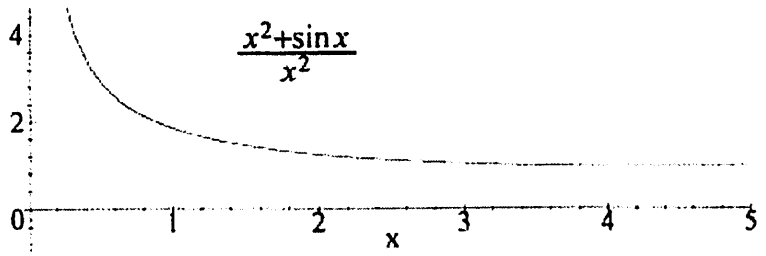
$$\lim_{x\to\infty} \frac{2x+\cos x}{2x} \quad \left(\frac{\infty}{\infty}\right) =$$

$$\lim_{x\to\infty} \frac{2-\sin x}{2} \quad \text{does not exist!}$$

"Why, yes, that is a very good use of L'Hopital, and it proves my point," you say. You started with a hard limit in one of the standard indeterminate forms, used L'Hopital twice, correctly I might add, and arrived at the answer, that $\lim_{x\to\infty} \frac{x^2+\sin x}{x^2}$ does not exist."

At that point the stranger laughs and laughs, and winds up saying between chortles, "Graph it, Sonny."

Fearlessly, you graph $\frac{x^2+\sin x}{x^2}$, and come up with this.



"Looks like the limit is 1 to me, boy. And last thing I heard was that 1 exists. Enjoy your johnnycake!"

You start to lose your appetite. Was L'Hopital lying all this time? Is the limit one? Or doesn't it exist? If "one" doesn't exist, then don't things start looking grim for 2, 3, and 4? Does it still take one to know one? Or is there a chance that the stranger made a mistake? Find the error.

Back to Find The Error!
Back to Doug Shaw's home page
Comment on this page

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Jim Mumaugh - Vaca High
From: Doug Shaw [doug@dougshaw.com]
Sent: Thursday, November 01, 2001 8:37 PM
To: Jim Mumaugh - Vaca High
Subject: Re: l'hopital rule

I was at your "Find the Error" site today looking for interesting things associated with l'hopital's rule in preparation for a beginning lecture on that topic. You have a problem there which I am unable to understand. The expression is

The solution is reading the rule carefully. It says that IF the lim of f'/g' exists, (and you are in 0/0 or inf/inf) THEN it is equal to the lim of f/g . It does NOT say that if the lim of f'/g' does not exist then the lim of f/g does not exist
Hope the problem teases your students in a good way!

← the key!!

Yes! Exactly!

.... and if $\lim_{x \to a} f'(x)/g'(x) = L$ then....

It is so easy to be led astray especially when the focus is on the very edges of topic.

Your problem & your assisting understanding of $ have helped me learn and as result, my students will ~~benefit~~ benefit. ~~benefit~~

Thank you so much.

Just because $\lim \frac{f'}{g'}$ DNE doesn't mean that $\lim \frac{f}{g}$

① $\lim_{x\to 0} \frac{xe^x}{e^x-1}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{xe^x+e^x}{e^x} = \lim_{x\to 0} x+1 = 1$

② $\lim_{x\to 1} \frac{e^x+1}{e^x-1} = \frac{e+1}{e-1}$

③ $\lim_{x\to 2} \frac{x^3-8}{x^2-4}$ $\frac{0}{0}$

$\lim_{x\to 2} \frac{3x^2}{2x} = \lim_{x\to 2} \frac{3}{2}x = 3$

④ $\lim_{x\to 2} \frac{x^3+8}{x^2+2} = \frac{16}{6} = \frac{8}{3}$

⑤ $\lim_{x\to 0} \frac{\sin x^2}{x\sin x}$ $\frac{0}{0}$ $= \lim_{x\to 0} \frac{2x\cos x^2}{\sin x + x\cos x}$ $\frac{0}{0}$

(u v in $x\sin x$; u v in $2x\cos x^2$)

$\lim_{x\to 0} \frac{2\cos x^2 + 4x^2\sin x^2}{\cos x + \cos x - x\sin x} = \frac{2-0}{2} = 1$

⑥ $\lim_{x\to \frac{\pi}{4}} \frac{\cos x^2}{\cos x} = \sqrt{2}\left(\cos\frac{\pi^2}{16}\right)$

⑦ $\lim_{x\to \frac{\pi}{2}} \frac{\sin x}{1+\cos x} = \frac{1}{1+0} = 1$

⑧ $\lim_{x\to 3^+} \frac{\ln(x-3)}{x-3}$ $\frac{-\infty}{0}$

11) counter assignment

$(1 + \text{nothing})^{\text{large}} = e.$

Revised:

change to $\lim_{x \to 0^+} \frac{\ln\left(\frac{\sin x}{x}\right)}{x}$

since $\lim \frac{\sin x}{x}$ is 1

we have $\ln 1$ is $\to 0$

over zero

Now we do have l'hopital situation.

Let $y = \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$

$\ln y = \frac{1}{x} \ln \frac{\sin x}{x}$

$\lim_{x \to 0^+} \frac{\ln \frac{\sin x}{x}}{x}$ " $\frac{0}{0}$

$\lim_{x \to 0} \frac{\frac{x\cos x - \sin x}{x^2}}{1}$ $\frac{0}{0}$

$\lim_{x \to 0^+} \frac{\ln \sin x - \ln x}{\ln x}$ $\frac{-\infty + \infty}{\infty}$?

$\lim_{x \to 0^+} \frac{\frac{\cos x}{\sin x} - \frac{1}{x}}{\frac{1}{x}}$ $\frac{\frac{1}{0} - \frac{1}{0}}{\frac{1}{0}}$?

$\lim_{x \to 0^+} \frac{\frac{x}{\sin x}\left[\frac{x\cos x - \sin x}{x^2}\right]}{1}$ $1 \cdot \left[\frac{0-0}{0}\right]$ STILL $\frac{0}{0}$

ARGh!

(9) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{-\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$

(10) $\lim_{x\to -\infty} \frac{5^x + 3^x}{4^x}$ $\frac{0}{0}$ $5^x = e^{x\ln 5}$

$\lim_{x\to -\infty} \frac{e^{x\ln 5} + e^{x\ln 3}}{e^{x\ln 4}} = \lim_{x\to -\infty} \frac{\ln 5\, e^{x\ln 5} + \ln 3\, e^{x\ln 3}}{\ln 4\, e^{x\ln 4}}$

$e^{x(\ln 5 - \ln 4)} + e^{x(\ln 3 - \ln 4)}$

$e^{x\ln\frac{5}{4}} + e^{x\ln\frac{3}{4}}$

$\lim_{x\to -\infty} \left(\frac{5}{4}\right)^x + \left(\frac{3}{4}\right)^x$

$0 \qquad +\infty$

let $y = (\ )^x$ etc.

after 2nd deriv $\frac{1}{4}$.

(11) $\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$ $\left(\frac{0}{0}\right)^{\infty}$

$\lim_{x\to 0^+} \left(\frac{\sin x}{x}\right)^{\frac{1}{x}} = 1$ $\lim_{x\to 0^+} \frac{\sin x}{x} = 1$

1 to infinity power = 1.

(12) $\lim_{x\to\infty} \frac{2^x}{3^x}$ $\frac{\infty}{\infty} \to 0$ because 3^x gets bigger than 2^x.

$\lim_{x\to\infty} \frac{e^{x\ln 2}}{e^{x\ln 3}} = \lim_{x\to\infty} e^{x(\ln 2 - \ln 3)} =$

since $\ln 2 - \ln 3 < 0$

$\frac{1}{e^{\text{Large}}} = 0.$

$$\lim_{x \to 0^+} \frac{\sin x}{1 - \sec^2 x}$$

$$\lim_{x \to 0^+} \frac{\sin x}{-\tan^2 x}$$

$$\frac{\sin x}{\frac{-\sin x}{\cos x}}$$

$$\lim_{x \to 0^+} \frac{-\cos^2 x}{\sin x}$$

$\frac{2-5\sin x}{3-7\cos x}$

(13) $\lim_{x\to\infty} \frac{2x+5\cos x}{3x-7\sin x}$ $\frac{\frac{1}{x}}{\frac{1}{x}}$ $\frac{0}{0}$

l'Hospital no help

$\lim_{x\to\infty} \frac{2+\frac{5}{x}\cos x}{3-\frac{7}{x}\sin x} = \frac{2}{3}$

(14) $\lim_{x\to\infty} \frac{3x^2+6\sqrt{x}}{x^2+8\sqrt{x}} = \lim_{x\to\infty} \frac{3+\frac{6}{x^{3/2}}}{1+\frac{8}{x^{3/2}}} = 3$

(15) $\lim_{x\to\infty} \frac{x^2+5}{2x^2+6x} = \lim_{x\to\infty} \frac{1+\frac{5}{x^2}}{2+\frac{6}{x}} = \frac{1}{2}$

(16) $\lim_{x\to 0^-} \frac{1-\cos x}{x+\tan x}$ $\frac{1-1}{0+0} = \frac{0}{0}$

$\lim_{x\to 0^-} \frac{\sin x}{1+\sec^2 x} = \frac{0}{2} = 0$

(17) $\lim_{x\to 0^+} \frac{1-\cos x}{x-\tan x} = \lim_{x\to 0^+} \frac{\sin x}{(1)-\sec^2 x}$

$\lim_{x\to 0^+} \frac{\cos x}{1-2\sec^2 x\tan x} = \lim_{x\to 0^+} \frac{-\cos^4 x}{-2\sin x}$ $-\infty$.

? (18) $\lim_{x\to\infty} \frac{e^{-x}}{x^2} = \lim_{x\to\infty} \frac{1}{x^2 e^x} = 0$ exponential on bottom.

(19) $\lim_{x\to\infty} \frac{\cos x}{x}$ $\cos x$ is at most 1 $\frac{1}{x}\to 0$, $\lim = 0$

Cos oscillates ∴ No limit

Still goes to zero Both ways

$$5^x = e^{\ln 5^x}$$

$$= e^{x \ln 5}$$

$$( )' = \frac{e^{x \ln 5} \ln 5}{5^x \ln 5}$$

(20) $\lim_{x\to\infty} \frac{\ln(x^2+1)}{\ln(x^2+8)}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{2x}{x^2+1} \cdot \frac{x^2+8}{2x} = \lim_{x\to\infty} \frac{2x}{2x} = 1$

(21) $\lim_{x\to 2} \frac{5^x+3^x}{x} = \frac{5^2+3^2}{2} = \frac{25+9}{2} = 17$

(22) $\lim_{x\to \frac{\pi}{2}} \frac{\sin 2x}{\sin 3x} = \frac{0}{-1} = 0$

(23) $\lim_{x\to 0} \frac{5^x-3^x}{x}$ $\frac{1-1}{0} = \frac{0}{0}$

$\lim_{x\to 0} \frac{e^{x\ln 5} - e^{x\ln 3}}{x} = \lim_{x\to 0} \ln 5\, e^{x\ln 5} - \ln 3\, e^{x\ln 3}$

$\ln 5 - \ln 3 = \ln \frac{5}{3}$.

(24) $\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} = 0$.

(25) $\lim_{x\to\infty} \left[(x^2-2x)^{\frac{1}{2}} - x\right] \frac{\left[\sqrt{x^2-2x}+x\right]}{\left[\sqrt{x^2-2x}+x\right]} =$

$\lim_{x\to\infty} \frac{x^2-2x-x^2}{\sqrt{x^2-2x}+x} = \lim_{x\to\infty} \frac{-2x}{\sqrt{x^2-2x}+x}$ Now $\frac{-\infty}{\infty}$

$\lim_{x\to\infty} \frac{-2}{\frac{2x-2}{2\sqrt{x^2-2x}}+1} = \lim_{x\to\infty} \frac{-2}{\frac{\frac{1}{x^2}\frac{x-1}{\sqrt{x^2-2x}}}{\frac{1}{\sqrt{x^2}}}+1}$

Notice $x-1 = \sqrt{(x-1)^2}$ so $\frac{x-1}{\sqrt{x^2-2x}} = \frac{\sqrt{(x-1)^2}}{\sqrt{x^2-2x}} = \sqrt{\frac{x^2-2x+1}{x^2-2x}}$

This has $\lim_{x\to\infty} \sqrt{\frac{x^2-2x+1}{x^2-2x}} = \lim_{x\to\infty} \sqrt{\frac{1-\frac{2}{x}+\frac{1}{x^2}}{1-\frac{2}{x}}} = 1$, so our limit is $\frac{-2}{2} = -1$

(26) $\lim_{x\to 0}\left(\frac{1}{1-\cos x}-\frac{2}{x^2}\right)=\lim_{x\to 0}\frac{x^2-2(1-\cos x)}{x^2(1-\cos x)}$

$\frac{0}{0}$

$$\lim_{x\to 0}\frac{2x-2\sin x}{2x(1-\cos x)+x^2\sin x}=$$

$$\lim_{x\to 0}\frac{2-2\cos x}{2(1-\cos x)+2x\sin x+2x\sin x+x^2\cos x}$$

$$\lim_{x\to 0}\frac{2\sin x}{2\sin x+4x\cos x+4\sin x+x^2\sin x+2x\cos x}$$

$$\lim_{x\to 0}\frac{2\cos x}{2\cos x+6x\sin x+6\cos x+4\cos x+x^2\cos x+2x\sin x}$$

$$\frac{2}{2+0+10}=\frac{2}{12}=\frac{1}{6}$$

(27) $\lim_{x\to 0}(1+3x)^{2/x}=\lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^6=e^6$

(28) $\lim_{x\to 0^+}x^{x^2}$ let $y=x^{x^2}$

$\ln y=x^2\ln x$

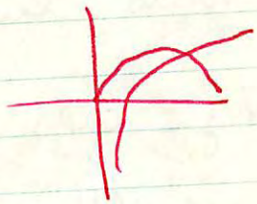
$$\lim_{x\to 0^+}\frac{\frac{1}{x}}{\frac{-2}{x^3}}=\lim_{x\to 0^+}\frac{1}{x}\cdot\frac{-x^3}{2}=0=\frac{\ln x}{\frac{1}{x^2}}\quad \begin{matrix}-\infty\\ +\infty\end{matrix}$$

So $\lim_{x\to 0}\ln y=0\Rightarrow y=e^0=1$

thus $\lim_{x\to 0^+}x^{x^2}=1$

(29) $\lim_{x\to 1^+} (x-1)\ln(x-1) = \lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{x-1}}$

$\lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{-1}{(x-1)^2}} = \lim_{x\to 1^+} -(x-1) = 0$



(30) $\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} =$ ($\ln x \to -\infty$, $\frac{1}{\sin x} \to \infty$)

$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} -\frac{\sin^2 x}{\cos x}\cdot\frac{1}{x} = \lim_{x\to 0^+} -\frac{\sin x}{x}\cdot\tan x = 0$ ($\sin x \to 0$, $\tan x \to 0$)

(31) $\lim_{x\to\infty} x\sin x = \lim_{a\to 0} \frac{\sin\frac{1}{a}}{a} = 1$ ~~Does not exist.~~ Oscillates.

(32) $\lim_{x\to 0} (\sin x)^x$ $\quad \ln y = x \ln \sin x$

$\frac{\ln \sin x}{\frac{1}{x}}$ $\quad \frac{-\infty}{\infty}$

$\lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{x}{\sin x}(-x\cos x) \Rightarrow 0 \therefore y = 1$

$\lim_{x\to 0} \ln y = 0 \therefore \lim_{x\to 0} y = 1$

(33) $\lim_{x\to 0} (\cos x)^{1/x}$ $\quad \ln y = \frac{1}{x}\ln\cos x = \frac{\ln\cos x}{x}$

$= \frac{0}{0}$

$\lim_{x\to 0} \frac{-\sin x}{1} = 0$ $\quad \lim_{x\to 0} \ln y = 0 \therefore \lim_{x\to 0} y = 1$

(34) $\lim_{x\to\frac{\pi}{2}^-} (1-\sin x)^{\cot x}$ ~~$\tan x$~~ $\cot x$

Let $y = (1-\sin x)^{\tan x}$

then $\ln y = \tan x \ln(1-\sin x) \Rightarrow -\infty$

Now $\ln y = \dfrac{\ln(1-\sin x)}{\cot x}$ Let $f(x) = \ln(1-\sin x)$ $\lim_{x\to\frac{\pi}{2}^-} f(x) = -\infty$

Let $g = \cot x$ $\lim_{x\to\frac{\pi}{2}^-} g(x) = \infty$ ~~Not L'Hopital's~~ Silentes.

So use l'hopital's rule.

So $f' = \dfrac{-\cos x}{1-\sin x}$ $g' = -\csc^2 x$ So

$\dfrac{f'}{g'} = \dfrac{\cos x \sin^2 x}{1-\sin x} \dfrac{1+\sin x}{1+\sin x} \Rightarrow \dfrac{\cos x \sin^2 x (1+\sin x)}{\cos^2 x} =$ etc

$\dfrac{\sin^2(1+\sin x)}{\cos x}$ Now $\lim_{x\to\frac{\pi}{2}^-} \dfrac{\sin^2(1+\sin x)}{\cos x} = \dfrac{1(2)}{\text{small}} \to 0$

So $\lim_{x\to\frac{\pi}{2}^-} \ln y = 0$ So $\lim_{x\to\frac{\pi}{2}^-} y = e^0 = 1$

Consider changing 34 to read

$\lim_{x\to\frac{\pi}{2}^-} (1+\cos x)^{\tan x}$ ie $(1+\text{small})^{\text{Large}}$ ie e

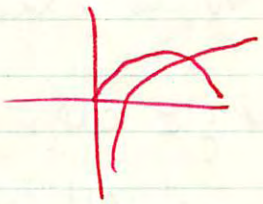
$\lim_{x\to\frac{\pi}{2}^-} \left[(1+\cos x)^{\frac{1}{\cos x}}\right]^{\sin x} = e^1 = e$

$\lim_{x\to\pi^-} (1-\sin x)^{\cot x} = \lim_{x\to\pi^-} \left[(1+(-\sin x))^{\frac{1}{-\sin x}}\right]^{-\cos x}$

$e^{-(-1)} = e$

(29) $\lim_{x\to 1^+} (x-1)\ln(x-1) = \lim_{x\to 1^+} \frac{\ln(x-1)}{\frac{1}{x-1}}$

$\lim_{x\to 1^+} \frac{\frac{1}{x-1}}{\frac{-1}{(x-1)^2}} = \lim_{x\to 1^+} -(x-1) = 0$



(30) $\lim_{x\to 0^+} \sin x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} =$ ($\ln x \to -\infty$, $\frac{1}{\sin x} \to \infty$)

$\lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}} = \lim_{x\to 0^+} -\frac{\sin^2 x}{\cos x}\cdot\frac{1}{x} = \lim_{x\to 0^+} -\frac{\sin x}{x}\cdot\tan x = 0$ ($\frac{\sin x}{x} \to 1$, $\tan x \to 0$)

(31) $\lim_{x\to\infty} x\sin x = \lim_{a\to 0} \frac{\sin\frac{1}{a}}{a} = 1$ ~~Does not exist. Oscillates.~~

(32) $\lim_{x\to 0} (\sin x)^x$ $\quad \ln y = x \ln \sin x$

$\lim_{x\to 0} \frac{\ln\sin x}{\frac{1}{x}}$ $\quad \frac{-\infty}{\infty}$

$\lim_{x\to 0} \frac{\frac{\cos x}{\sin x}}{\frac{-1}{x^2}} = \lim_{x\to 0} \frac{x}{\sin x}(-x\cos x) \Rightarrow 0 \therefore y = 1$

$\lim_{x\to 0} \ln y = 0 \therefore \lim_{x\to 0} y = 1$

(33) $\lim_{x\to 0} (\cos x)^{1/x}$ $\quad \ln y = \frac{1}{x}\ln\cos x = \frac{\ln\cos x}{x}$

$= \frac{0}{0}$

$\lim_{x\to 0} \frac{-\sin x}{1} = 0$

$\lim_{x\to 0} \ln y = 0 \therefore \lim_{x\to 0} y = 1$

(34) $\lim_{x\to\frac{\pi}{2}^-} (1-\sin x)^{\cot x}$ ~~$\tan x$~~ $\cot x$ Let $y = (1-\sin x)^{\tan x}$

then $\ln y = \tan x \ln(1-\sin x) \Rightarrow -\infty$

Now $\ln y = \dfrac{\ln(1-\sin x)}{\cot x}$ Let $f(x) = \ln(1-\sin x)$ $\lim_{x\to\frac{\pi}{2}^-} f(x) = -\infty$

Let $g = \cot x$ $\lim_{x\to\frac{\pi}{2}^-} g(x) = \infty$ ~~Not Needed~~ Indeterminate.

So use l'Hopital's rule.

So $f' = \dfrac{-\cos x}{1-\sin x}$ $g' = -\csc^2 x$ So

$\dfrac{f'}{g'} = \dfrac{\cos x \sin^2 x}{1-\sin x} \dfrac{1+\sin x}{1+\sin x} \Rightarrow \dfrac{\cos x \sin^2 x (1+\sin x)}{\cos^2 x} =$ etc

$\dfrac{\sin^2(1+\sin x)}{\cos x}$ Now $\lim_{x\to\frac{\pi}{2}^-} \dfrac{\sin^2(1+\sin x)}{\cos x} = \dfrac{1(2)}{\text{small}} \to \infty$

So $\lim_{x\to\frac{\pi}{2}^-} \ln y = 0$ So $\lim_{x\to\frac{\pi}{2}^-} y = e^0 = 1$

Consider changing 34 to read

$\lim_{x\to\frac{\pi}{2}^-} (1+\cos x)^{\tan x}$ ie $(1+\text{small})^{\text{Large}}$ ie e

$\lim_{x\to\frac{\pi}{2}^-} \left[(1+\cos x)^{\frac{1}{\cos x}}\right]^{\sin x} = e^1 = e$

$\lim_{x\to\pi^-} (1-\sin x)^{\cot x} = \lim_{x\to\pi^-} \left[(1+(-\sin x))^{\frac{1}{-\sin x}}\right]^{-\cos x}$

$e^{-(-1)} = e$

(34) with change of exponent

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)^{\cot x} \qquad \text{let } y=\ln(1-\sin x)^{\cot x}$$

$$\lim_{x\to\frac{\pi}{2}^-}\frac{\ln(1-\sin x)}{\tan x} \quad \frac{-\infty}{-\infty} \quad \text{apply L'Hopital's Rule}$$

$$\lim_{x\to\frac{\pi}{2}^-}\frac{-\cos x}{(1-\sin x)\sec^2 x} = \lim_{x\to\frac{\pi}{2}^-}\frac{-\cos^3 x}{1-\sin x} \quad \frac{0}{0}$$

$$\lim_{x\to\frac{\pi}{2}^-}\frac{+3\cos^2 x\sin x}{-\cos x} = \lim_{x\to\frac{\pi}{2}^-} -3\cos x\sin x = -3\cdot 1\cdot 0 = 0$$

So $\lim_{x\to\frac{\pi}{2}^-}\ln y = 0 \Rightarrow y \Rightarrow e^0 \therefore y = 1$

$$\lim_{x\to\frac{\pi}{2}^-}(1-\sin x)^{\cot x} = 1$$

(35) $\lim_{x\to\infty} x^{\frac{1}{x}} = 1$ $\quad \ln y = \frac{\ln x}{x}$ $\quad \lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln x}{x} = 0$

$\therefore \ln(y) \Rightarrow 0 \Rightarrow y \Rightarrow 1$

? (36) $\lim_{x\to\infty}(2^x - x^{10})^{\frac{1}{x}}$

1

For large x $2^x - x^{10}$ becomes 2^x

$$(2^x)^{\frac{1}{x}} = 2.$$

(39) $\lim_{x\to\infty} \frac{(x^3-x^2-4x)^{1/3}}{x} = \lim_{x\to\infty} \left(\frac{x^3-x^2-4x}{x^3}\right)^{1/3} =$

$\lim_{x\to\infty} \left(1-\frac{1}{x}-\frac{4}{x^2}\right)^{1/3} = 1$

(40) $\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{\frac{1}{x}}$ let $y = \left(\frac{1+2^x}{2}\right)^{\frac{1}{x}}$ then

$\ln y = \frac{1}{x}\ln\left(\frac{1+2^x}{2}\right) = \frac{\ln\left(\frac{1+2^x}{2}\right)}{x}$

$\frac{1+2^x}{2}$

$\frac{1}{2}+\frac{1}{2}2^x$

$\frac{1}{2}\cdot 2^x \ln 2$

$\lim_{x\to 0} \lim \left(\frac{1+2^x}{2}\right) = 0 = \lim_{x\to 0} x$ so use l'Hopital's Rule.

if $f = \ln\frac{1+2^x}{2}$ then $f' = \frac{1}{\frac{1+2^x}{2}} \cdot \frac{1}{2} \cdot 2^x \ln 2 = \frac{2^x \ln 2}{1+2^x}$

So $\lim_{x\to 0} f' = \frac{\ln 2}{2}$

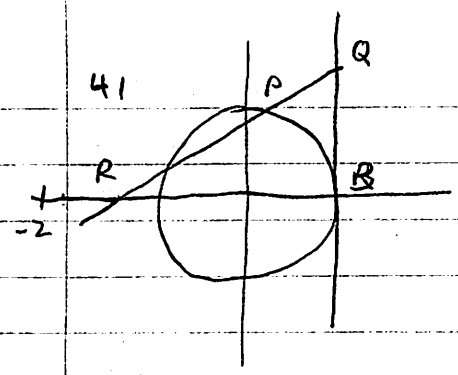
if $g = x$ then $g' = 1$ So

$\lim_{x\to 0} \ln y = \frac{\ln 2}{2}$ but $\frac{\ln 2}{2} = \ln\sqrt{2}$

$\therefore \lim_{x\to 0} y = \sqrt{2}$ but $y = \left(\frac{1+2^x}{2}\right)^{\frac{1}{x}}$ so

$\lim_{x\to 0} \left(\frac{1+2^x}{2}\right)^{\frac{1}{x}} = \sqrt{2}$

41



$\overline{RQ} = \overline{BP}$

6.Q Chapt 6.

(1) a) f is continuous over $[a,b]$, differentiable over (a,b), $f(a) = f(b)$

b) $\exists\ \overline{X} \in (a,b) \Rightarrow f'(\overline{X}) = 0$

(2) a) f is continuous over $[a,b]$, differentiable over (a,b)

b) $\exists\ \overline{X} \in (a,b) \Rightarrow f'(\overline{X}) = \frac{f(b)-f(a)}{b-a}$

(3) a) f is decreasing, x intercept is $(a,0)$
b) max of f at $a, f(a)$
c) inflection pt at f

(5a) $f(x) = \tan x - x \qquad 0 \le x < \frac{\pi}{2}$

WTS $f' > 0$

$f'(x) = \sec^2 x - 1 > 0 \ \forall x \ \therefore f' > 0.$

b) $f(x) = \tan x - x$

$f(0) = 0$

$\forall x > 0 \quad f(x) > f(0) = 0$

$\tan x - x > 0 \Rightarrow \tan x > x$

C $\frac{\sin x}{\cos x} - x > 0$

$\sin x - x\cos x > 0$

$x\cos x - \sin x < 0 \quad \forall x\ (0, \frac{\pi}{2})$

D WTS $\frac{\sin x}{x}$ is decreasing. Let $f(x) = \frac{\sin x}{x}$

$f'(x) = \frac{x\cos x - \sin x}{x^2}.$

by 5C $x\cos x - \sin x < 0. \quad x^2 \neq 0 \ \therefore f' < 0 \ \forall x\ (0, \frac{\pi}{2})$

$\therefore$ f is a decreasing fctn.

(7) $(1.001)^x \quad \log_{10} x \quad x^{100} \quad 2^x$

$\log_{10} x < x^{100} < (1.001)^x < 2^x \quad$ as $x \to \infty$

$\lim_{x\to\infty} (1.001)^x =$ $\quad \ln Y = x \ln 1.001$

~~$(1.001)^x = x^{100}$~~ $\quad$ ~~$(1.001)^x < 2^x$~~

$\lim \dfrac{x^{100}}{(1.001)^x} = 0$ because base is variable power gets bigger faster than variable to fixed power

(10) a) $2, 5\frac{1}{2}$ b) $-\frac{1}{3}, 4, 6$ c) $-1, 4, 5\frac{1}{2}, 6$
D) No global max e) ~~-3~~ -1

(11) Given $\dfrac{dY}{dx} = 3Y^4$ WTS $\dfrac{d^2Y}{dx^2} = 36Y^7$

$\dfrac{d^2Y}{dx^2} = \dfrac{d\left(\frac{dY}{dx}\right)}{dx} = \dfrac{d(3Y^4)}{dx} = 12Y^3 \dfrac{dY}{dx}$ but $\dfrac{dY}{dx} = 3Y^4$

$\therefore \quad 12Y^3 \cdot 3Y^4 = 36Y^7 = \dfrac{d^2Y}{dx}$

redo

(12) $m > 0$ WTP $(1+x)^m > 1+mx \quad \forall x > 0$.

Let $f(x) = (1+x)^m - (1+mx)$ WTP $f(x) > 0 \; \forall x$.

$f(0) = 0 \quad f'(x) = m(1+x)^{m-1} - m$

$m\left((1+x)^{m-1} - 1\right)$

$m(x^{m-1})$

No PS!

Sel Next

$f'(x) > 0 \; \forall x \;\therefore f$ is increasing $\therefore f(x) > 0 \; \forall x$.

$\therefore \; 1 + x^m - (1+mx) > 0 \Rightarrow 1 + x^m > 1 + mx.$

$2^m - 1 - m$ pg 242

(12) $(1+x)^m > (1+mx)$ $m>0$ $x>0$

Let $f(x) = (1+x)^m - (1+mx)$ WTP $f(x) > 0$ $\forall x>0, m>0$

$f(0) = 1^m - 1 = 0$.

$f'(x) = m(1+x)^{m-1} - m$

$= m\left((1+x)^{m-1} - 1\right)$ $f' > 0$ for $m>1$ obvious.

$f(1) = 2^m - 2m$ & $f' > 0$. $\therefore$ f continues to be >0.

$\therefore$ Discussion is centred on $0<m<1$

Let $x = .1$, $m = .1$

$(1+x)^m > (1+mx)$

$(1.1)^{.1} > (1+(.1)(.1)$

$1.009576 > 1.01$ false!!

Pblm should read $m>1$ $x>0$

100 sist on Pumbers lins for f, f' and f''

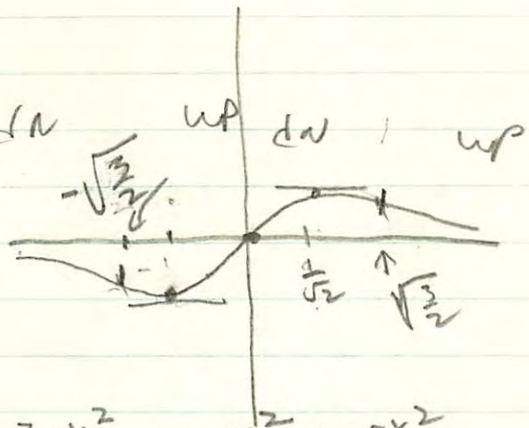
(13) $y = xe^{-x^2}$

$y' = -2x^2 e^{-x^2} + e^{-x^2}$

$y' = -e^{-x^2}(2x^2 - 1)$

$y' \overset{set}{=} 0 \Rightarrow x = \pm \frac{1}{\sqrt{2}}$

$y = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}} \approx \frac{1}{2}$



$y'' = 4x^3 e^{-x^2} - 4xe^{-x^2} - 2xe^{-x^2}$

$e^{x^2}(4x^3 - 6x) \overset{set}{=} 0$

$2x(2x^2 - 3)$

$x = \pm\sqrt{\frac{3}{2}}, 0$

$y'' =$

$\lim_{x \to \infty} xe^{-x^2} = 0$

$\lim_{x \to -\infty} xe^{-x^2} = 0$

(19) a) $\lim_{x \to 0} \frac{e^{x^2} - 1}{1 - \cos x} = \lim_{x \to 0} \frac{2xe^{x^2}}{\sin x} = \lim_{x \to 0} \frac{2e^{x^2} + 4x^2 e^{x^2}}{\cos x} = 2$

b) $\lim_{x \to 0} \frac{x - \tan^{-1} x}{x - \sin x} = \lim_{x \to 0} \frac{1 - \frac{1}{x^2+1}}{1 - \cos x} =$

$\lim_{x \to 0} \frac{\frac{2x}{(x^2+1)^2}}{\sin x} = \lim_{x \to 0} \frac{2x}{(x^2+1)^2} \cdot \frac{1}{\sin x} =$

$\lim_{x \to 0} \frac{2}{(x^2+1)^2} \cdot \frac{x}{\sin x} = 2$

c) $\lim_{x \to 0} x^{1-\cos x}$ $\quad \ln y = 1 - \cos x \ln x$

Too much $\lim_{x \to 0} \frac{\ln x}{\frac{1}{1-\cos x}}$

$\lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-\sin x}{(1-\cos x)^2}} = \lim_{x \to 0} \frac{-1}{x} \cdot \frac{(1-\cos x)^2}{\sin x}$

$\lim_{x \to 0} -2(1-\cos x)(\sin^2 x)x +$

(20) a) $f(x) = \csc 5x = \frac{1}{\sin 5x}$

$f'(x) = \frac{-5\cos 5x}{\sin^2 5x} = -5\cot 5x \csc 5x$

$f''(x) =$

① a) $e^{\ln 2} = 2$ $\ln e^8 = 8$ $\tan^{-1}(\tan \frac{\pi}{3}) = \frac{\pi}{3}$

b) $\sin \frac{3\pi}{2} = -1$ $\tan \frac{\pi}{6} = \sqrt{3}$ $\cos \frac{\pi}{3} = \frac{1}{2}$

c) $\sin^{-1} \frac{1}{2} = \frac{\pi}{6}$ $\tan^{-1}(-1) = -\frac{\pi}{4}$ $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$

② a) 1 rad $\approx 57°$

b) $\frac{\theta R^2}{2}$ c) $\lim_{x \to 0} (1+x)^{\frac{1}{x}}$ d) 2,718

③ $y = f(x)$

$f(3) = 5$ $f'(3) = 4$ find $\left.\frac{d(y^2)}{dx}\right]_{x=3}$

$2y \frac{dy}{dx}$ $2 \cdot 5 \cdot 4 = 40$.

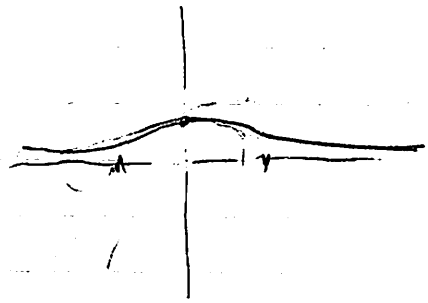
---

④ $y = e^{-x^2}$

$y' = -2xe^{-x^2}$ $y > 0$ $x < 0$

$y'' = 4x^2 e^{-x^2} - 2e^{-x^2}$

$2e^{-x^2}(2x^2 - 1)$



⑤ a) $= e^{3x}$ $3e^{3x}$ b) 10^{-5x} $e^{-5x \ln 10}$

$-5 \ln 10 e^{-5x \ln 10}$

$(-5 \ln 10) 10^{-5x}$

c) $x^2 \sin 3x$

$2x \sin 3x + 3x^2 \cos 3x$

d) $\frac{\cos 5x}{x^2}$ $\frac{-5x^2 \sin 5x - 2x \cos 5x}{x^4}$

e) $x^{5/6} \sin^{-1} x$

$\frac{5}{6} x^{-1/6} \sin^{-1} x + \frac{x^{5/6}}{\sqrt{1-x^2}}$

f) $\cot x^2$

$\frac{\cos x^2}{\sin x^2}$

$\frac{-2x \sin^2 x^2 - 2x \cos^2 x^2}{\sin^2 x^2} = \frac{-2x}{\sin^2 x^2}$

6 a) $x^5 - 2x + \ln(2x+3)$

$5x^4 - 2 + \frac{2}{2x+3}$

b) $\operatorname{Sec}^{-1} 3x$

$\frac{3}{|3x|\sqrt{9x^2-1}} = \frac{1}{|x|\sqrt{9x^2-1}}$

c) $(x^7 + x^2)^{10}$

$10(x^7+x^2)^9(7x^6+2x)$

D) $\frac{x^2}{1+3x}$ $\frac{2x(1+3x) - 3x^2}{(1+3x)^2} =$

$\frac{2x + 3x^2}{(1+3x)^2}$

(7) a) $\ln|x + \sqrt{x^2+25}|$ $x + \sqrt{x^2+25} > 0$ because $x = \sqrt{x^2}$

$\frac{1 + \frac{x}{\sqrt{x^2+25}}}{x + \sqrt{x^2+25}} = \frac{\frac{x + \sqrt{x^2+25}}{\sqrt{x^2+25}}}{x + \sqrt{x^2+25}} = \frac{1}{\sqrt{x^2+25}}$ $\sqrt{x^2} < \sqrt{x^2+25}$

$\therefore x < \sqrt{x^2+25}$

b) $(x-3)\sqrt{6x-x^2} + 9\sin^{-1}\left(\frac{x-3}{3}\right)$ $\frac{x}{3} - 1$

$0 < \sqrt{x^2+25} - x$

$\sqrt{6x-x^2} + \frac{(x-3)\left(\frac{3}{6} - \frac{1}{2}x\right)}{\sqrt{6x-x^2}} + \frac{9 \cdot \frac{1}{3}}{\sqrt{1 - \left(\frac{x-3}{3}\right)^2}}$

$\frac{6x - x^2 - (x-3)^2}{\sqrt{6x-x^2}} + \frac{9}{\sqrt{9 - (x-3)^2}}$ $4 - x^2 + 6x - 9$

$\frac{6x - x^2 - x^2 + 6x - 9 + 9}{\sqrt{6x-x^2}} = \frac{-2x^2 + 12x}{\sqrt{6x-x^2}} = \frac{2(-x^2+6x)}{\sqrt{6x-x^2}} =$

$2\sqrt{6x-x^2}$

(8) a) $\frac{3}{1+x^2}$ $3\tan^{-1}x$ b) $\frac{5}{\sqrt{1-x^2}}$ $5\sin^{-1}x$

c) $6e^{-x}$ $-6e^{-x}$ d) x^4 $\frac{x^5}{5}$

e) $\frac{1}{\sqrt{1-4x^2}}$ $\frac{1}{2}\sin^{-1}2x$ f) $\frac{1}{x^2}$ $-\frac{1}{x}$

g) $\frac{1}{x}$ $\ln x$ h) $\frac{x}{\sqrt{1-x^2}}$ $-(1-x^2)^{\frac{1}{2}}$

(9) $3x^2+2x$ b $\cos x$ c $\cos 2x$

a) $x^3+\frac{x^2}{2}$ $\sin x$ $\frac{1}{2}\sin 2x$

d) $\frac{1}{\sqrt{x}}$ $2\sqrt{x}$ e) $\frac{1}{2x+1}$ $\frac{1}{2}\ln(2x+1)$

(11) a) $L=0$ b) $L=2$ c) NO d) $0^{\infty}=0$ ~~N.D.~~

e) NO f) $\infty^{-\infty}\times 0$ $\frac{1}{\infty^{\infty}}=0$

(12) a) 0 b) 0 c) ND d) ND

(13) a) f, g differentiable on (a,b) $g'(x)\neq 0$ for any $x\in(a,b)$ $\lim_{x\to a} f(x)=0$ $\lim_{x\to a} g(x)=0$

$\lim_{x\to a}\frac{f'(x)}{g'(x)}=L.$

b) $\lim_{x\to a}\frac{f(x)}{g(x)}=L$

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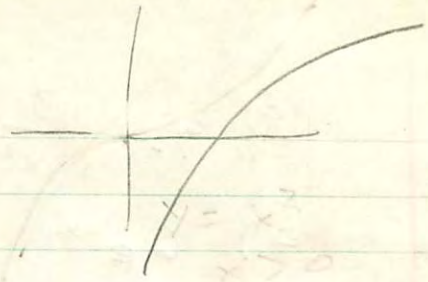
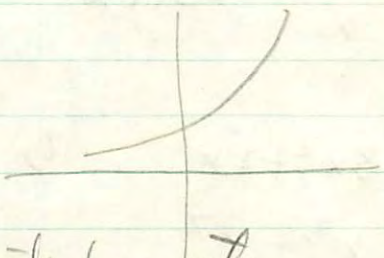
14) a $f'(x) > 0 \quad f''(x) < 0 \quad \forall x$



b) $f > 0 \quad f'' < 0$ No concave down eventually will cross x axis.

c) $f(x) > 0 \quad f'(x) > 0$



15a) Locate critical values of x
find $f(c)$ & $f(\text{endpts})$. Pick largest.

b) $f' \quad f''$

Try

$$\lim_{x \to 1} x^{\frac{1}{\log_2 x}} = \lim_{x \to 1} x^{\log_x 2} = 2$$

$$\text{if } 1^{\infty} = 2 \text{ ??!}$$

$$y = x^{\frac{1}{\log_2 x}}$$

$$\log_2 y = \frac{1}{\log_2 x} \log_2 x$$

$$\log_2 y = 1 \quad \therefore \quad y = 2^1 = 2$$

$$\text{thus } \lim_{x \to \infty} x^{\frac{1}{\log_2 x}} = \lim_{x \to \infty} 2 = 2.$$

(3) a) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x}{x} = \frac{\sqrt{2}}{2}\cdot\frac{4}{\pi} = \frac{2\sqrt{2}}{\pi}$

b) $\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \frac{\sqrt{2}}{2}}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\cos x}{1} = \frac{\sqrt{2}}{2}$

c) $\lim_{h\to 0} \frac{e^{3+h}-e^3}{1-h} = 0$

D) $\lim_{h\to 0} \frac{e^{3+h}-e^3}{h} = \lim_{h\to 0} \frac{e^{x+h}-e^x}{h}\Big]_{x=3} = e^3$

E) $\lim_{x\to 0} (1+3x)^{\frac{1}{x}} = \lim_{x\to 0}\left[(1+3x)^{\frac{1}{3x}}\right]^3 = e^3$

f) $\lim_{x\to 1} \frac{\sin \pi x}{x-1} = \lim_{x\to 1} \pi\cos \pi x = \pi\cos\pi = -\pi$

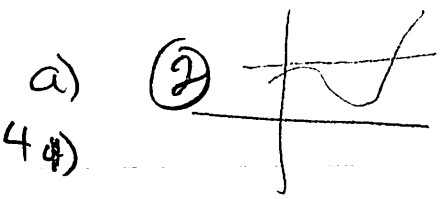
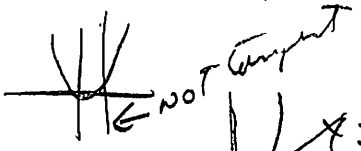
g) $\lim_{x\to\infty} x^{\frac{1}{\log_2 x}} = \lim_{x\to\infty} x^{\frac{\log_2}{\log x}}$

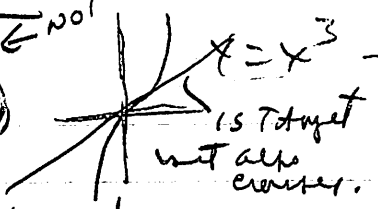
$\log_2 y = \frac{1}{\log_2 x}\log_2 x = \frac{\log_2 x}{\log_2 x} = 1$

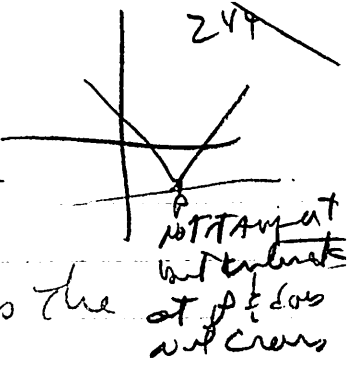
$\lim_{x\to\infty} \log_2 y = 1 \therefore \lim_{x\to\infty} 2^1 = 2$

h) $\lim_{x\to 0} \frac{\cos\sqrt{x}-1}{\tan x} = \lim_{x\to 0} \frac{\frac{-1}{2\sqrt{x}}\sin\sqrt{x}}{\sec^2 x} =$

$\lim_{x\to 0} -\frac{\sin\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{2}\cdot\frac{1}{\sec^2 x} = -\frac{1}{2}$

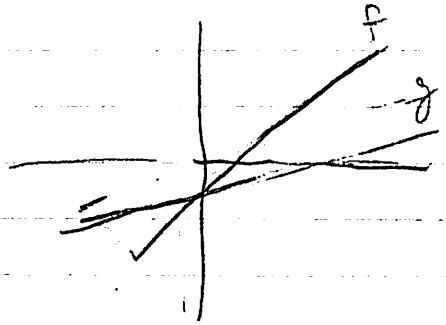
4) a) ②  or 

① 



b) Slope of the curve is the same as the line tangent at that pt.

c) 1st der of f(x) when f(x) changes location.

⑤ 

a) not @ x

b) yes @ x > 0

⑥ $f(0) = 0$ $f'(x) \geq 1$ (at least 1)

$f(3) \geq 3$

a) $\frac{1}{x^2+1}$ b) $\frac{x}{x^2+1}$

$\frac{-2x}{(x^2+1)^2}$ $\frac{x^2+1-2x^2}{(x^2+1)^2}$

$\frac{1-x^2}{(x^2+1)^2}$

(10) $\frac{1}{5}\tan^{-1}\frac{x}{5}$ b) $\sin^{-1}\frac{x}{5}$

$\frac{1}{5}\frac{1}{5/}\frac{25}{25+x^2}$ $\frac{1}{5/}\frac{5}{x^2-25}$

$\frac{1}{25+x^2}$ $\frac{1}{x^2-25}$

(11) a) 10^x

$e^{x\ln 10}$

$(\ln 10)10^x$

b) $\log_{10} x$

$\frac{\ln x}{\ln 10}$

$\frac{1}{x\ln 10}$

(12) a) $\frac{(3x+1)^4}{12}$

$\frac{3\cdot 4(3x+1)^3}{12}$

$(3x+1)^3$

b) $\frac{12}{(3x+1)^4}$

$\frac{(-12)(3)(4)(3x+1)^3}{(3x+1)^8}$

$\frac{-144}{(3x+1)^5}$

Review. 1–6

① a) $\sqrt{x}$ b) $\frac{1}{\sqrt{x}}$ ② x^3 $\frac{1}{x^3}$

$\frac{1}{2\sqrt{x}}$ $-\frac{1}{2x^{3/2}}$ $3x^2$ $-\frac{3}{x^4}$

③ $\frac{x}{3} - \frac{4}{9}\ln(3x+4)$

$\frac{1}{3} - \frac{4 \cdot 3}{9(3x+4)}$

$\frac{3x+4-4}{3(3x+4)} = \frac{x}{3x+4}$

④ a) e^{x^2} b) $\frac{e^{x^2}}{2x}$

$2xe^{x^2}$ e^{x^2}

5 a) $e^{\sqrt{x}}$ b) $2e^{\sqrt{x}}(\sqrt{x}-1)$

$\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ $\frac{e^{\sqrt{x}}(\sqrt{x}-1)2e^{\sqrt{x}}}{2\sqrt{x}} + \frac{2e^{\sqrt{x}}}{2\sqrt{x}}$

$\frac{e^{\sqrt{x}}(\sqrt{x}-1+1)}{\sqrt{x}} = e^{\sqrt{x}}$

6 a) x^2e^{3x} b) x^2e^{-3x}

$6x^2e^{3x} + 2xe^{3x}$ $-6xe^{-3x} + 2xe^{-3x}$

7 a) $\sin^2 2x$ b) $\sin^3 2x$

$4\sin 2x \cos 2x$ $6\sin^2 2x \cos 2x$.

⑧ $\ln(\sec 3x + \tan 3x)$ b) $\ln \cos 3x$

$\frac{3\sec 3x \tan 3x + 3\sec^2 3x}{\sec 3x + \tan 3x}$ $\frac{-3\sin 3x}{\cos 3x}$

$\frac{3\sec 3x(\tan 3x + \sec 3x)}{(\sec 3x + \tan 3x)}$ $-3\tan 3x$.

$3\sec 3x$

(13) $x^2 \ln x$

$2x \ln x + x$

b) $\dfrac{\ln x}{x^2}$

$\dfrac{x - 2x\ln x}{x^4}$

$\dfrac{1 - 2\ln x}{x^3}$

(14) $\frac{x}{3} - \frac{4}{9}\ln|3x+4|$

$\frac{1}{3} - \frac{4}{9}\cdot\frac{3}{|3x+4|}$

$\frac{3x+4-4}{3|3x+4|} = \frac{x}{3x+4}$

(15) $x - 2\ln(x+1) - \frac{1}{x+1}$

$1 - \frac{2}{x+1} + \frac{1}{(x+1)^2}$

$\frac{(x+1)^2 - 2(x+1) + 1}{(x+1)^2} = \frac{x^2+2x+1-2x-2+1}{(x+1)^2}$

$\frac{x^2}{(x+1)^2}$

(16) $\frac{1}{8}\left[\ln(2x+1) + \frac{2}{2x+1} - \frac{1}{2(2x+1)}\right]$

$\frac{1}{8}\left[\ln(2x+1) + \frac{3}{4x+2}\right]$

$\frac{1}{8}\left[\frac{2}{2x+1} - \frac{12}{(4x+2)^2}\right] = \frac{1}{8}\left[\frac{2}{2x+1} - \frac{12}{4(2x+1)^2}\right]$

$\frac{1}{8}\left[\frac{16x+8-12}{4(2x+1)^2}\right] = \frac{1}{8}\left[\frac{16x-4}{4(2x+1)^2}\right] = \frac{1}{8}\left[\frac{4x-1}{(2x+1)^2}\right]$

17 $\sqrt{\frac{2}{15}(5x+7)^3}$

$\left(\frac{2}{15}\right)^{1/2} \cdot (5x+7)^{3/2}$

$\frac{2^{1/2}}{15^{1/2}} \cdot \frac{3}{2}(5x+7)^{1/2} \cdot 5$

$\frac{\sqrt{2}}{\sqrt{15}} \cdot \frac{15}{2}(5x+7)^{1/2} = \sqrt{\frac{15(5x+7)}{2}}$

(18) a) $x \cos 5x + 5 \sin 5x$

$\cos 5x - 5x \sin 5x + 5\cos 5x$

$6\cos 5x - 5x \sin 5x$,

b) $\cos^2 5x$

$-10 \cos 5x \sin 5x$

(19) $\ln \sqrt[3]{x}$

$\frac{1}{3}\ln x$

$\frac{1}{3x}$

(20) $\ln \sqrt{4+x}\sqrt[3]{x^2+1}$

$\frac{1}{2}\ln 4+x + \frac{1}{3}\ln x^2+1$

$\frac{1}{2(4+x)} + \frac{2x}{3(x^2+1)}$

(21) $\ln \frac{10^x}{\sin x} = \ln 10^x - \ln \sin x$

$x \ln 10 - \ln \sin x$

$\ln 10 - \cot x$

(22) $\ln(1+\cos x)^2 = 2\ln(1+\cos x)$

$\frac{-2\sin x}{1+\cos x}$.

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(23) $\lim_{x\to 0} \frac{\sin 2x}{e^{3x}-1} = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2x}{e^{3x}-1}$

$\lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{2x}{e^{3x}-1}$

$1 \qquad \lim_{x\to 0} \frac{2}{3e^{3x}} = \frac{2}{3}$

(24) $\lim_{x\to\infty} \frac{(x^2+1)^5}{e^x} = 0$ "e^x on bottom." Polynomial on top.

(25) $\lim_{x\to\infty} \frac{3x-\sin x}{x+\sqrt{x}} = \lim_{x\to\infty} \frac{3-\frac{\sin x}{x}}{1+\frac{1}{\sqrt{x}}} = 3$

(26) $\lim_{x\to 0} \frac{1-\cos^2 3x}{x^2+x^3} = \lim_{x\to 0} \frac{\sin^2 3x}{3x} \cdot \frac{\sin 3x}{3x} \cdot \frac{9}{1+x}$

$\lim_{x\to 0} \frac{\sin 3x}{3x} \cdot \frac{\sin 3x}{3x} \cdot \frac{9}{1+x} = 9,$

$\lim_{x\to 0} \frac{3\sin 3x \cos 3x}{2x+3x^2} = \lim_{x\to 0} \frac{-9\sin^2 3x + 9\cos^2 3x}{2+6x}$

(9)

$\lim \frac{6\cos 3x \sin 3x}{2x+3x^2} \qquad \frac{3\sin 6x}{2x+3x^2}$

$\frac{18\cos 6x}{2+6x} = 9$

(33) $\exists\, x \in (c,d)$
Law of the mean guarantees $f(x) = \dfrac{F(d)-F(c)}{d-c}$
which yields $f(x)(d-c) = f(d)-f(c)$

(34) a) 2 b) 1, 2 c) 1, 2, 3

(44) $f(x) = x^3 + ax^2 + c \qquad a \neq 0,\ c \neq 0$
$f(0) = c$ which is positive
$\lim_{x \to -\infty} f(x) = -\infty$ $\therefore$ there is at least 1 neg. root.
by intermediate value thm. Now look at $f'(x)$.
$f'(x) = 3x^2 + 2ax = x(3x+2a) = x\left(x + \frac{2a}{3}\right) \overset{\text{set}}{=} 0$
$x = 0$ or $x = \frac{-2a}{3}$ since $a < 0$ $-\frac{2a}{3} > 0$, so
f' does not have ANY negative zeros. ie if there were a 2nd Neg Root f' would have to be zero for some $x < 0$. Since f' has no zeros for $x < 0$ there cannot be 2 (or more) Neg Roots. $\therefore$ there is exactly 1 Neg Root.

(46)

x	y	$x+y$	$\lvert x\rvert + \lvert y\rvert$	$\lvert x+y\rvert$
2	3	5	5	5
-2	3	1	5	1
-2	-3	-5	5	5
2	-3	-1	5	1

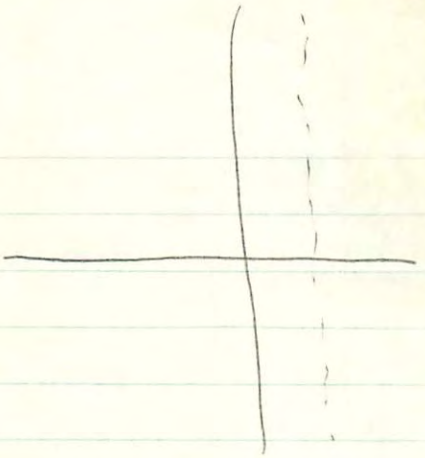
b) $|x| + |y| \geq |x+y|$
c) obvious.

(55) $y = \frac{1}{x^2} + \frac{1}{x-1}$

$y = \frac{x-1+x^2}{x^2(x-1)}$

$y =$

$y' = -\frac{2}{x^3} - \frac{1}{(x-1)^2}$



$\frac{-2(x^2-2x+1)-x^3}{x^3(x-1)^2} = \frac{-(x^3+2x^2-4x+2)}{x^3(x-1)^2}$

(68) a) Yes every chord is // to a tangent to f.
(.) Lawof the mean.
b) NO Consider $y = x^3$ at $x = 0$.

85 (a) $f(x) = \frac{b}{a}\sqrt{x^2-a^2}$ a, b constant

If $y = cx+d$ is asymptote then $f'(x)$ has same slope (or approaches it) for very + or − x.
And $\lim_{x\to\infty} f(x) - y = 0$

$f'(x) = \dfrac{bx}{a\sqrt{x^2-a^2}}$

$$\lim_{x\to\infty} \frac{bx}{a\sqrt{x^2-a^2}} = \boxed{\lim_{x\to\infty} \frac{b}{a\sqrt{1-\frac{a^2}{x^2}}}} = \frac{b}{a} \quad \therefore c = \frac{b}{a}$$

$\lim_{x\to\infty} \left[\frac{b}{a}\sqrt{x^2-a^2} - (cx+D)\right] \overset{\text{set}}{=} 0$ (The difference between the fcn & the asymptote must go to zero)

$\lim_{x\to\infty} \left\{\frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x - D\right\} = 0 \quad \therefore$

$D = \lim_{x\to\infty} \frac{b}{a}\sqrt{x^2-a^2} - \frac{b}{a}x \overset{\text{must}}{=} 0$

So $c = \frac{b}{a}$ $d = 0$ $\therefore$ asymptote must be $y = \frac{a}{b}x$.

* If $\lim_{x\to-\infty}$ we get: $\lim_{x\to-\infty} \dfrac{bx}{a\sqrt{x^2-a^2}} =$

$$\lim_{x\to-\infty} \frac{bx}{-a\,x\sqrt{1-\frac{a^2}{x^2}}} = -\frac{b}{a}$$

"Four Corners of the World"

For each problem, select the proper multiple choice response. Include with your work a detailed explanation of your method and justification.

1. Find the value c which satisfies the Mean Value Theorem for $f(x) = \ln x$ over $1 \le x \le 4$.

a) $\frac{\ln 4}{3}$ b) $\frac{3}{\ln 4}$ c) $\ln 4$ d) $3 \ln 4$ e) $\frac{4}{3}$

2. If $y = \sqrt{x}^{\sqrt{x}}$ then $\ln y =$

a) $\frac{x \ln x}{2}$ b) $\frac{\sqrt{x} \ln x}{2}$ c) $\frac{x \ln x}{4}$ d) $\frac{\ln x}{4}$ e) $\sqrt{x} \ln x$

3. If $y = \cos^2 x - \sin^2 x$ then $\frac{d^2y}{dx^2} =$

a) 0 b) $4 \cos x \sin x$ c) $-2 \sin 2x$ d) $4 \sin 2x$ e) $-4 \cos 2x$

4. If $y = 3x^{100}$ then $\frac{d^{100}y}{dx^{100}} =$

a) $3(100!)$ b) $3(100!)x$ c) 0 d) $300x^{99}$ e) $300!$

5. If $f(x) = \sqrt{x^3}$ then $f''''\left(\frac{1}{4}\right) =$

a) -3 b) $\frac{-3}{64}$ c) $\frac{9}{64}$ d) $\frac{9}{32}$ e) $\frac{3}{8}$

6. If $f(x) = (\sin x)^2$, then $\frac{d^2y}{dx^2} =$

a) $2 \cos x$ b) $\cos 2x$ c) $\sin^2 x$ d) $\cos^2 x$ e) $2 \cos 2x$

7. If $f(x) = \frac{1}{x}$, then the sixth derivative of $f(x)$ is:

a) $\frac{6}{x^6}$ b) $\frac{6!}{x^7}$ c) $\frac{1}{x^6}$ d) $-\frac{6}{x^7}$ e) 0

8. The absolute minimum value of $f(x) = \frac{3x-1}{x+2}$ on $[0,2]$ is:

a) 0 b) 2 c) $-\frac{1}{2}$ d) $\frac{5}{4}$ e) -2

9. The x-coordinate of the absolute maximum of $f(x) = 3x^3 - \frac{9}{2}x^2$ on $[-2,2]$ is:

a) 0 b) 2 c) 1 d) -2 e) $\frac{1}{2}$

10. Find the absolute maximum value of $f(x) = 2x^3 + 3x^2 - 12x + 1$ on the interval $[-1,2]$

a) 19 b) 14 c) -6 d) 5 e) 21

11. $\lim_{x\to\infty}\left(3+2e^{x}\right)^{3/x}=$

a) 3 b) $3e$ c) 0 d) e^3 e) Nonexistent

12. For the function $f(x)=|x-4|^2$, find the value of $f''(c)$ where c is a critical number of the function.

a) 0 b) 2 c) 4 d) 3 e) $f''(c)$ does not exist

13. Find all values of x where $f(x)$ is increasing if $f(x)=\left(2x^2-8\right)^{4/3}$.

a) $(-2,0)\cup(2,\infty)$ b) $(-\infty,-2)\cup(2,\infty)$ c) $(-\infty,0)\cup(2,\infty)$ d) $(-2,2)$ e) $(0,\infty)$

14. For $f(x)=x^3+ax^2$ to have a relative minimum at $x=1$ then $a=$

a) $-\frac{3}{2}$ b) -1 c) $-\frac{2}{3}$ d) $\frac{2}{3}$ e)1

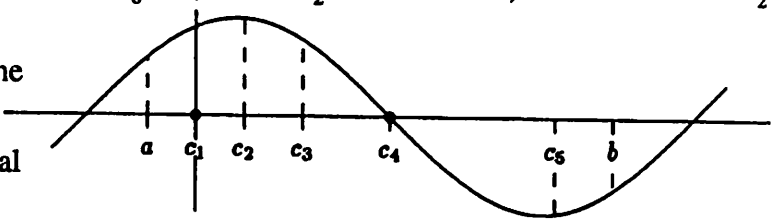
15. The greatest slope of $f(x)=x^3-9x^2$ for $0\le x\le 10$ occurs at $x=$

a) 0 b) 2 c) 3 d) 6 e) 10

16. If $f(x)=x^{1/3}-x^{2/3}$ then f has relative extrema points at:

a) $x=0$ only b) $x=\frac{1}{8}$ only c) $x=0$ and $x=\frac{1}{8}$ d) $x=\frac{1}{2}$ e) $x=0$ and $x=\frac{1}{2}$

17. Which value of c_i, most closely approximates the value of c satisfying the conclusion of the Mean Value Theorem for the function f over the interval $a\le x\le b$.



a) c_1 b) c_2 c) c_3 d) c_4 e) c_5

18. For $f(x)=x^{2/3}-1$ on $[-1,1]$ the "c" value that satisfies Rolle's Theorem is:

a) -1 b) 1 c) 1 or -1

d) There is no c value because $f(-1)$ is not zero.

e) There is no c value because f is not differentiable on $(-1, 1)$.

19) For the following functions

I. $f(x)=x^3+1$ on $[-1,1]$ II. $f(x)=1-|x|$ on $[-1,1]$ III. $f(x)=x^2-x$ on $[0,1]$

Rolle's Theorem holds for :

a) I only b) II only c) III only d) I and II only e) I and III only

20) The Mean Value Theorem does not apply to $f(x)=|3x-2|$ on $[0,1]$ because:

a) $f(0)\neq f(1)$ b) $f(0)>f(1)$

c) $f(x)$ is not continuous on [0, 1].

d) $f(x)$ is not differentiable on (0, 1).

e) $\lim_{x\to 2/3} f(x)=f(\frac{2}{3})$

"Four Corners of the World"

For each problem, select the proper multiple choice response. Include with your work a detailed explanation of your method and justification.

1. Find the value c which satisfies the Mean Value Theorem for $f(x) = \ln x$ over $1 \le x \le 4$.

a) $\frac{\ln 4}{3}$ (b) $\frac{3}{\ln 4}$ c) $\ln 4$ d) $3 \ln 4$ e) $\frac{4}{3}$

2. If $y = \sqrt{x}^{\sqrt{x}}$ then $\ln y =$

a) $\frac{x \ln x}{2}$ (b) $\frac{\sqrt{x} \ln x}{2}$ c) $\frac{x \ln x}{4}$ d) $\frac{\ln x}{4}$ e) $\sqrt{x} \ln x$

3. If $y = \cos^2 x - \sin^2 x$ then $\frac{d^2y}{dx^2} =$

a) 0 b) $4 \cos x \sin x$ c) $-2 \sin 2x$ d) $4 \sin 2x$ (e) $-4 \cos 2x$

4. If $y = 3x^{100}$ then $\frac{d^{100}y}{dx^{100}} =$

(a) $3(100!)$ b) $3(100!)x$ c) 0 d) $300x^{99}$ e) 300!

5. If $f(x) = \sqrt{x^3}$ then $f'''\left(\frac{1}{4}\right) =$

(a) -3 b) $\frac{-3}{64}$ c) $\frac{9}{64}$ d) $\frac{9}{32}$ e) $\frac{3}{8}$

6. If $f(x) = (\sin x)^2$, then $\frac{d^2y}{dx^2} =$

a) $2 \cos x$ b) $\cos 2x$ c) $\sin^2 x$ d) $\cos^2 x$ (e) $2 \cos 2x$

7. If $f(x) = \frac{1}{x}$, then the sixth derivative of $f(x)$ is:

a) $\frac{6}{x^6}$ (b) $\frac{6!}{x^7}$ c) $\frac{1}{x^6}$ d) $-\frac{6}{x^7}$ e) 0

8. The absolute minimum value of $f(x) = \frac{3x-1}{x+2}$ on $[0,2]$ is:

a) 0 b) 2 (c) $-\frac{1}{2}$ d) $\frac{5}{4}$ e) -2

9. The x-coordinate of the absolute maximum of $f(x) = 3x^3 - \frac{9}{2}x^2$ on $[-2,2]$ is:

a) 0 (b) 2 c) 1 d) -2 e) $\frac{1}{2}$

10. Find the absolute maximum value of $f(x) = 2x^3 + 3x^2 - 12x + 1$ on the interval $[-1,2]$

a) 19 (b) 14 c) -6 d) 5 e) 21

11. $\lim_{x\to\infty}\left(3+2e^{x}\right)^{3/x} =$

a) 3 b) $3e$ c) 0 d) e^3 e) Nonexistent

12. For the function $f(x)=|x-4|^2$, find the value of $f''(c)$ where c is a critical number of the function.

a) 0 b) 2 c) 4 d) 3 e) $f''(c)$ does not exist

13. Find all values of x where $f(x)$ is increasing if $f(x)=\left(2x^2-8\right)^{4/3}$.

a) $(-2,0)\cup(2,\infty)$ b) $(-\infty,-2)\cup(2,\infty)$ c) $(-\infty,0)\cup(2,\infty)$ d) $(-2,2)$ e) $(0,\infty)$

14. For $f(x)=x^3+ax^2$ to have a relative minimum at $x=1$ then $a=$

a) $-\frac{3}{2}$ b) -1 c) $-\frac{2}{3}$ d) $\frac{2}{3}$ e)1

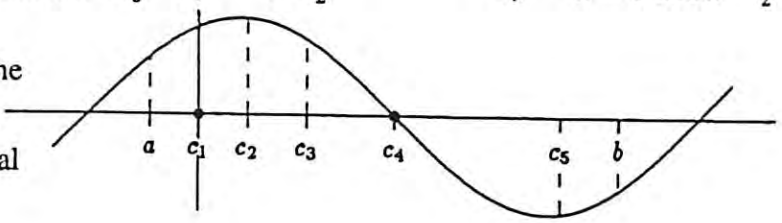
15. The greatest slope of $f(x)=x^3-9x^2$ for $0\le x\le 10$ occurs at $x=$

a) 0 b) 2 c) 3 d) 6 e) 10

16. If $f(x)=x^{1/3}-x^{2/3}$ then f has relative extrema points at:

a) $x=0$ only b) $x=\frac{1}{8}$ only c) $x=0$ and $x=\frac{1}{8}$ d) $x=\frac{1}{2}$ e) $x=0$ and $x=\frac{1}{2}$

17. Which value of c_i, most closely approximates the value of c satisfying the conclusion of the Mean Value Theorem for the function f over the interval $a\le x\le b$.



a) c_1 b) c_2 c) c_3 d) c_4 e) c_5

18. For $f(x)=x^{2/3}-1$ on $[-1,1]$ the "c" value that satisfies Rolle's Theorem is:

a) -1 b) 1 c) 1 or -1

d) There is no c value because $f(-1)$ is not zero.

e) There is no c value because f is not differentiable on $(-1, 1)$.

19) For the following functions

I. $f(x)=x^3+1$ on $[-1,1]$ II. $f(x)=1-|x|$ on $[-1,1]$ III. $f(x)=x^2-x$ on $[0,1]$

Rolle's Theorem holds for :

a) I only b) II only c) III only d) I and II only e) I and III only

20) The Mean Value Theorem does not apply to $f(x)=|3x-2|$ on $[0,1]$ because:

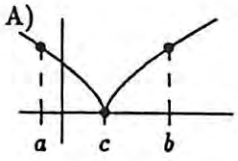
a) $f(0)\neq f(1)$ b) $f(0)>f(1)$

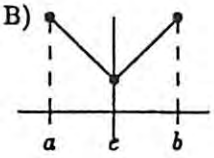
c) $f(x)$ is not continuous on [0, 1].

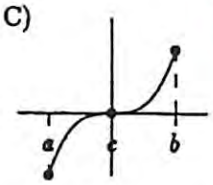
d) $f(x)$ is not differentiable on (0, 1).

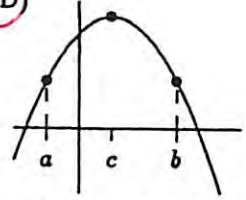
e) $\lim_{x\to 2/3} f(x)=f(\frac{2}{3})$

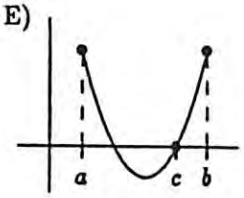
21. Which picture meets the conditions of Rolle's Theorem over an interval [a, b] and which indicates a value c meeting the conclusion of Rolle's Theorem.

A) 

B) 

C) 

D) 

E) 

22. Find the number(s) for "c" which satisfy the conditions of the Mean Value Theorem of differential calculus for $f(x) = x^3 - 2x^2$ on $[0, 2]$.

a) 0 b) $\frac{4}{3}$ c) $0, \frac{4}{3}$ d) $\frac{4}{3}, 2$ e) $\frac{2}{3}$

23. Given $f(x) = -\frac{1}{2}x^4 + \frac{1}{3}x^3 + 3x^2 - 4$. Find the intervals on which f is increasing.

a) $-\frac{3}{2} < x < 0$ or $x > 2$ b) $x < -\frac{3}{2}$ or $x > 2$ c) $-\frac{3}{2} < x < 2$
d) $x < -\frac{3}{2}$ or $0 < x < 2$ e) $x > 2$

24. Given $H'(x) = (x-1)(x+2)(x-3)^2$. Determine the x coordinates of the relative maximum points of $H(x)$.

a) 1 b) -2 c) 3 d) 1, 3 e) $-2, 3$

25. Let $f(x) = 3bx^2 + ax - 5$ and let $(2, -3)$ be a relative extremum of f, then the values of a and b respectively are:

a) $2, -3$ b) $2, -6$ c) $2, -\frac{1}{6}$ d) $-6, 2$ e) $-2, 6$

26. The total number of relative extrema of the function H whose derivative for all x is given by $H'(x) = 3x(x-5)(2x-3)^2(3x-4)^4$ is:

a) 0 b) 1 c) 2 d) 3 e) 4

27. If $f(x) = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 \le x \le 5 \\ x - 5 & \text{if } x > 5 \\ -x - 5 & \text{if } x < -5 \end{cases}$ then f is increasing on

a) $(5, \infty)$ b) $(-\infty, \infty)$ c) $(-5, 0) \cup (5, \infty)$ d) $(-\infty, -5) \cup (0, 5)$ e) $(-5, 5)$

28. If $f(x) = \begin{cases} 4 - x^2 & \text{if } |x| < 2 \\ |x| - 2 & \text{if } |x| \ge 2 \end{cases}$ then f has minimum values when $x =$

a) 0 b) 2 c) 1 or -1 d) 2 or -2 e) -2

29) The equation of the tangent to the curve $y = x^3 - x^2 + 2x$ at its point of inflection is:

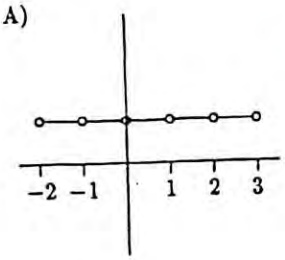
a) $y = 2x$ b) $23x - y = 45$ c) $45x - 27y = -1$ d) $9x - 27y = -13$ e) $63x - 27y = 5$

30) If $f(x) = 3x^5 - 10x^4 + 7x - 5$, then there are inflection points when $x =$

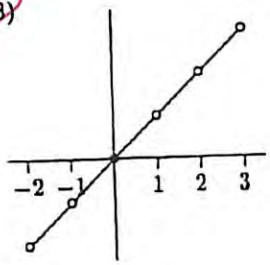
a) 0 b) 2 c) 0 or 2 d) -2 e) 0 or -2

31. Which graph of f best suits the description of f over $-2 \le x \le 3$.
$f'(n)$ is undefined for n any natural number.
$f'(x) = 1$ for any $x \neq n$.

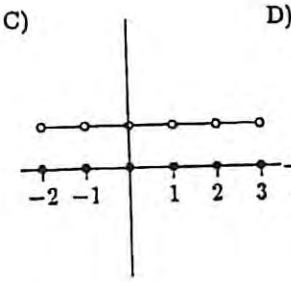
A)



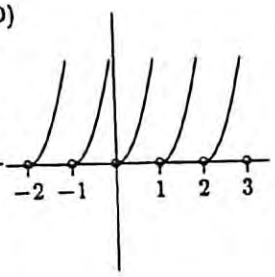
B)



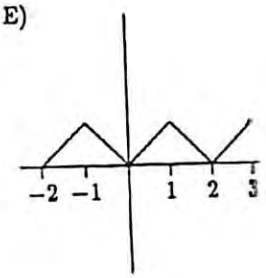
C)



D)

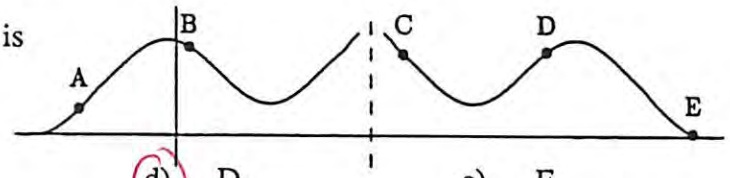


E)



32. At which of the five points on the graph on the right is

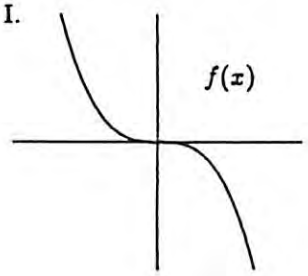
$$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} < 0$$



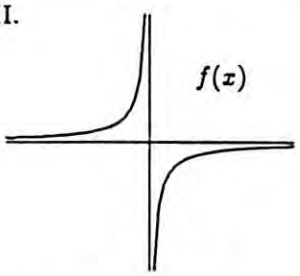
a) A b) B c) C d) D e) E

33. For which of the following functions is $f'(x) < 0$ whenever $f(x)$ defined?

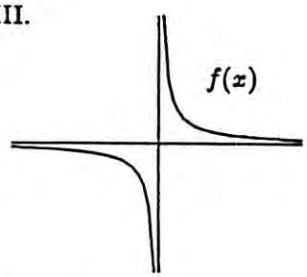
I.



II.

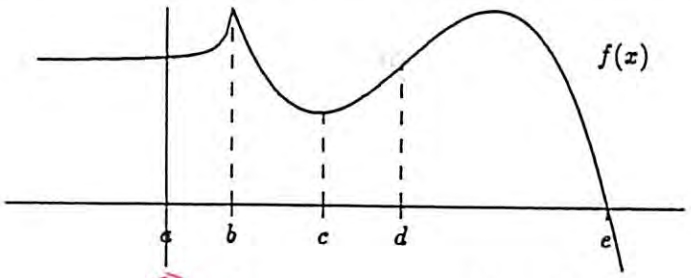


III.



a) I only b) II only c) III only d) I and II only e) I and III only

34. On the graph of f show on the right, at which indicated x-value does it appear that the rate of change in the slope of f is zero? At x =

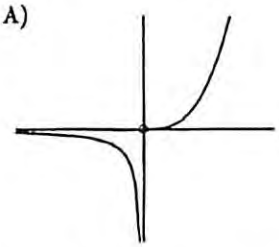


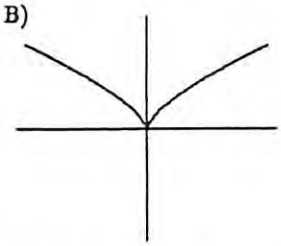
a) A b) B c) C d) D e) E

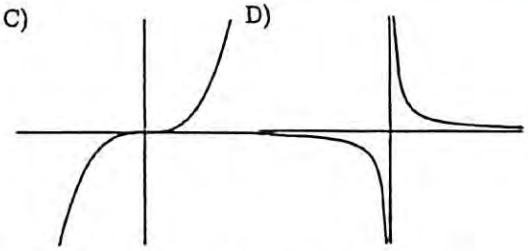
35. If $f(x) = x^4 + 4x^3 + 6x^2 + 4x$, the inflection point(s) is(are):
a) $(-1, 0)$ b) $(1, 16)$ c) $(0, 1)$ d) There are none. e) $(1, 16)$ and $(-1, 0)$

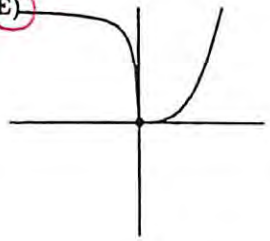
36. If $f(x) = \dfrac{x^2 - 1}{x^2}$, then the graph of f is concave up for:
a) All values of x b) $-1 < x < 1$ c) $x < -1$ or $x > 1$ d) No values of x e) $-\sqrt{2} < x < \sqrt{2}$

37. Which graph of f suits the description of f, a continuous function, where
$f(x) > 0,\ f'(x) > 0,\ f''(x) > 0$ for $x > 0$ **and** $f(x) > 0,\ f'(x) < 0,\ f''(x) < 0$ for $x < 0$.

A) 

B) 

C) D) 

E) 

38. If f is a continuous function with $f''(x) = -5(3+2x)(2-3x)^2(3x-1)^3$, find the set of values of x for which f has an inflection point.

a) $\{-\frac{3}{2}, \frac{1}{3}, \frac{2}{3}\}$ b) $\{0, -\frac{3}{2}, \frac{1}{3}, \frac{2}{3}\}$ c) $\{-\frac{3}{2}, \frac{1}{3}\}$ d) $\{-\frac{3}{2}\}$ e) $\{\frac{1}{3}\}$

39) Let $f(x) = 2x^3 + 3x^2 - 36x$. Which of these choices is necessary to makethe point (2, – 44) is a minimum?

I. $f'(2) = 0$ II. $f''(2) < 0$ II. $f(2) < 0$

a) I only b) II only c) III only d) I and II only e) I, II, and III

40. The relative minimum of $g(x) = \dfrac{3x^2+3}{x}$ is $x = 1$ because:

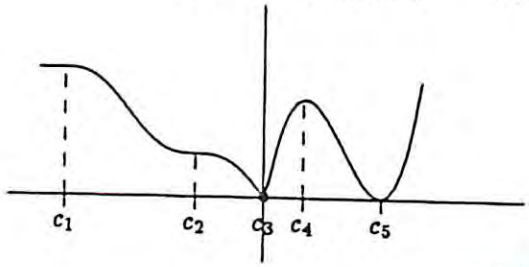
I. $g'(1) = 0$ II. $g''(1) > 0$ III. $g(1) = 6$

a) I only b) II only c) III only d) I and II only e) I, II, and III

41. If $f(x) = \dfrac{x^2+1}{x}$, then f has a relative minimum when $x =$

a) 0 b) 1 c) -1 d) no minimum e) $\sqrt{3}$

42. On the graph, which indicated x-value would satisfy: $f''(x) > 0$ when $f'(x) = 0$?



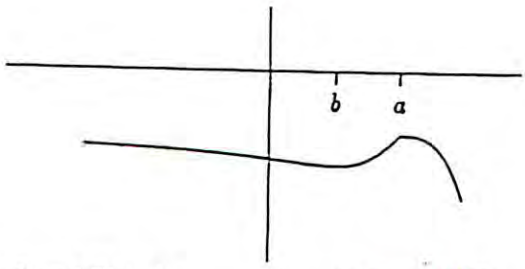
a) c_1 b) c_2 c) c_3 d) c_4 e) c_5

43. Given the graph of f, which of the following statements are true?

I. $f''(x) < 0$ when $x > a$.

II. $f'(x) < 0$ when $x > b$.

III. $f'(a) = 0$.



a) I only b) II only c) III only d) I and II only e) I, II, and III

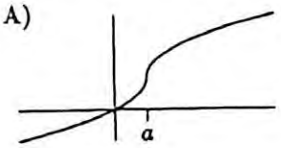
44. For the function $f(x) = \sqrt[3]{x}$, find $\lim_{x \to \infty} f'(x)$.

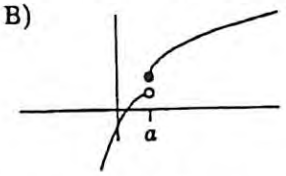
a) ∞ b) 0 c) 1 d) 3 e) 2

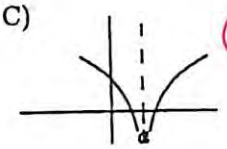
45. The function $f(x) = \dfrac{x^3 - 2x + 1}{(1-x)^2}$ has an oblique asymptote:

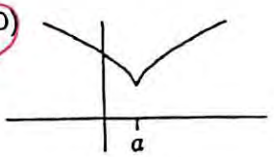
a) $y = x$ b) $y = x - 2$ c) $y = x + 2$ d) $y = 2 - x$ e) $y = x - 3$

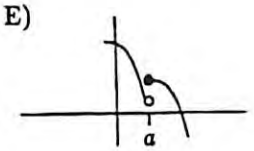
46. If f is continuous $\forall$ x and differentiable for all x except $x = a$ and always concave downward, then which of the following could be a graph for f?

A) 

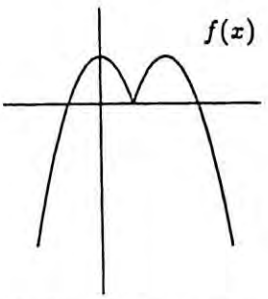
B) 

C) 

D) 

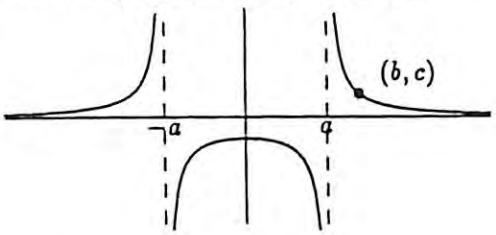
E) 

47. For the graph of f, which of the following statements is true?



a) $f(x)$ is continuous.
b) $f(x)$ is differentiable for all x.
c) $f(x) = |f(x)|$
d) $f(x)$ is an even function.
e) $f(x)$ has no relative minimum point.

48. For the function f with the given graph, which of the following statements is not true?

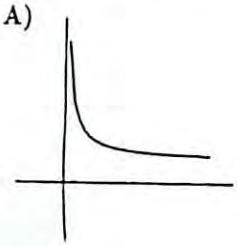


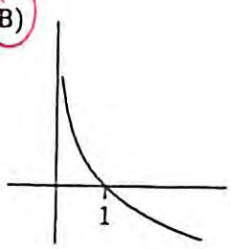
a) $f(x)$ is discontinuous at $-a$ and a.
b) $f'(x)$ is positive if $x < 0$.
c) There is a relative maximum at $x = 0$.
d) $f''(x)$ is positive if $-a < x < a$.
e) $\lim_{x \to -b} f(x) = c$.

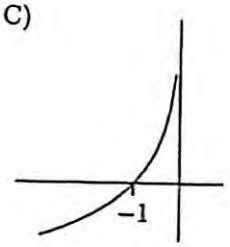
49. $\lim_{x \to \infty} \ln\left(1 + \dfrac{1}{x}\right)^{2x} =$

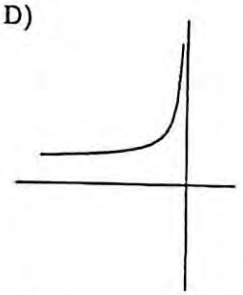
a) 2 b) ∞ c) 1 d) $2e$ e) e^2

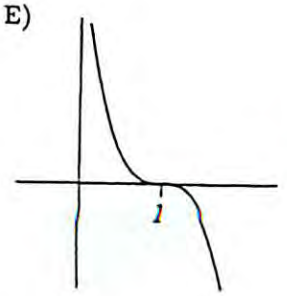
50., The graph of $y = \ln\left(\dfrac{1}{x}\right)$ is :

A) 

B) 

C) 

D) 

E) 

51. $\lim_{x \to \pi/4} \dfrac{\sin\left(x - \frac{\pi}{4}\right)}{x^2 - \pi^2/16} =$

a) 0 b) $\frac{2}{\pi}$ c) $\frac{1}{8\pi}$ d) $\frac{8}{3\pi}$ e) Nonexistent

52. $\lim_{x \to \infty} \dfrac{\ln\left(x^3\right)}{e^{2x}} =$

a) 0 b) ∞ c) 1 d) $\frac{1}{2}$ e) $\frac{3}{2}$

53. In the invalid argument:

$$\lim_{x \to 0} \frac{x^6 - x^5 + x^4 - x^3}{x^5 - x^4 + x^3 - x^2} = \lim_{x \to 0} \frac{6x^5 - 5x^4 + 4x^3 - 3x^2}{5x^4 - 4x^3 + 3x^2 - 2x} \quad \textbf{Step 1}$$

$$\lim_{x \to 0} \frac{30x^4 - 20x^3 + 12x^2 - 6x}{20x^3 - 12x^2 + 6x - 2} \quad \textbf{Step 2}$$

$$\lim_{x \to 0} \frac{120x^3 - 60x^2 + 24x - 6}{60x^2 - 24x + 6} \quad \textbf{Step 3}$$

$$\lim_{x \to 0} \frac{360x^2 - 120x + 24}{120x - 24} \quad \textbf{Step 4}$$

$$\lim_{x \to 0} \frac{720x - 120}{120} \quad \textbf{Step 5}$$

At what step should one stop using L'Hôpital's Rule and evaluate the limit? Step:

a) 1 b) 2 c) 3 d) 4 e) 5

54. Evaluate $\lim_{x \to \frac{\pi}{2}^-} \left(\dfrac{\sin^2 x}{1 - \cos x}\right)^{\tan x}$

a) 0 b) 1 c) $\frac{\pi}{2}$ d) ∞ e) e

55. Evaluate $\lim_{x \to 0} \dfrac{x - \tan x}{\sin x - x}$

a) 2 b) 0 c) ∞ d) 1 e) $\frac{1}{2}$

56. Evaluate $\lim_{x \to \infty} \dfrac{e^x}{x^2 + 4}$

a) There is no limit b) $\frac{1}{4}$ c) 0 d) 1 e) $\frac{1}{2}$

57. $\lim_{x \to 0} \dfrac{\sin^{-1} x}{\tan^{-1} 2x}$

a) ∞ b) $-\infty$ c) 0 d) 1 e) $\frac{1}{2}$

58. $\lim_{x \to \infty} x^{1/x} =$

a) ∞ b) 1 c) e d) $\frac{1}{e}$ e) 0

59. Evaluate $\lim_{x \uparrow \pi/2} \dfrac{\sec x}{\tan x}$.

a) ∞ b) $-\infty$ c) 1 d) -1 e) 0

60. Evaluate $\lim_{x \to \infty} [\ln x - \ln(1+x)]$.

a) There is no limit. b) $-\infty$ c) -2 d) 0 e) 1

61. $\lim_{x \to \infty} \left(\sqrt{x^2 + 2x} - x\right) =$

a) 0 b) $\frac{1}{2}$ c) 1 d) 2 e) The limit does not exist.

62. $D_x\left[\ln \sec^{-1} x^2\right] =$

a) $\dfrac{1}{\sec^{-1} x^2}$ b) $\dfrac{2x}{\sec^{-1} x^2}$

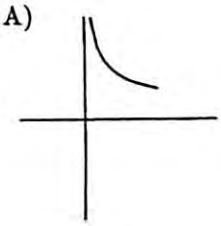
c) $\dfrac{1}{\left(\sec^{-1} x^2\right)\left(x^2\sqrt{x^4 - 1}\right)}$ d) $\dfrac{2}{x\sqrt{x^2 - 1}\left(\sec^{-1} x^2\right)}$ e) $\dfrac{2}{x\sqrt{x^4 - 1}\left(\sec^{-1} x^2\right)}$

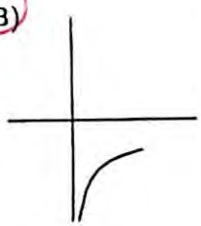
63. $\dfrac{d}{dx}\left[\tan 3^{2x}\right] =$

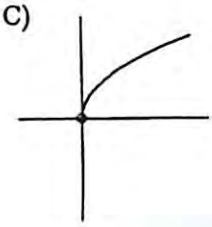
a) $\sec^2 3^{2x}$ b) $\sec^2 3^{2x} \ln 3$ c) $\left(\sec^2 3^{2x}\right)3^{2x}$

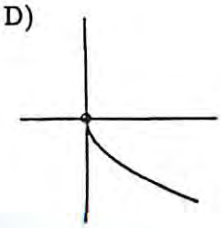
d) $2\left(3^{2x}\right)\left(\sec^2 3^{2x}\right)\ln 3$ e) $\left(3^{2x}\right)\left(\sec^2 3^{2x}\right)\ln 3$

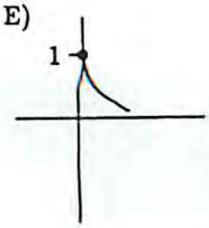
64. $\lim_{x \to 0^+} \dfrac{\ln x}{x}$ behaves graphically near zero like:

A) 

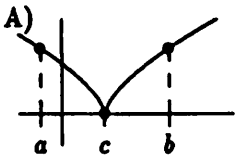
B) 

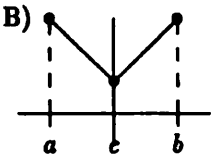
C) 

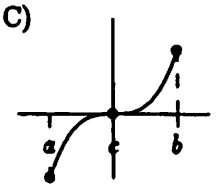
D) 

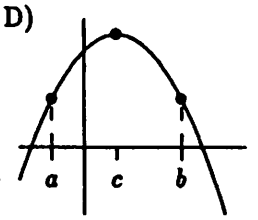
E) 

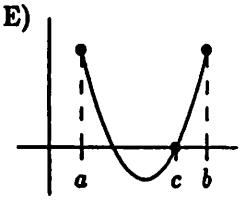
21. Which picture meets the conditions of Rolle's Theorem over an interval [a, b] and which indicates a value c meeting the conclusion of Rolle's Theorem.

A) 

B) 

C) 

D) 

E) 

22. Find the number(s) for "c" which satisfy the conditions of the Mean Value Theorem of differential calculus for $f(x) = x^3 - 2x^2$ on $[0, 2]$.

a) 0 b) $\frac{4}{3}$ c) $0, \frac{4}{3}$ d) $\frac{4}{3}, 2$ e) $\frac{2}{3}$

23. Given $f(x) = -\frac{1}{2}x^4 + \frac{1}{3}x^3 + 3x^2 - 4$. Find the intervals on which f is increasing.

a) $-\frac{3}{2} < x < 0$ or $x > 2$ b) $x < -\frac{3}{2}$ or $x > 2$ c) $-\frac{3}{2} < x < 2$
d) $x < -\frac{3}{2}$ or $0 < x < 2$ e) $x > 2$

24. Given $H'(x) = (x-1)(x+2)(x-3)^2$. Determine the x coordinates of the relative maximum points of $H(x)$.

a) 1 b) -2 c) 3 d) 1, 3 e) $-2, 3$

25. Let $f(x) = 3bx^2 + ax - 5$ and let $(2, -3)$ be a relative extremum of f, then the values of a and b respectively are:

a) $2, -3$ b) $2, -6$ c) $2, -\frac{1}{6}$ d) $-6, 2$ e) $-2, 6$

26. The total number of relative extrema of the function H whose derivative for all x is given by $H'(x) = 3x(x-5)(2x-3)^2(3x-4)^4$ is:

a) 0 b) 1 c) 2 d) 3 e) 4

27. If $f(x) = \begin{cases} \sqrt{25 - x^2} & \text{if } -5 \le x \le 5 \\ x - 5 & \text{if } x > 5 \\ -x - 5 & \text{if } x < 5 \end{cases}$ then f is increasing on

a) $(5, \infty)$ b) $(-\infty, \infty)$ c) $(-5, 0) \cup (5, \infty)$ d) $(-\infty, -5) \cup (0, 5)$ e) $(-5, 5)$

28. If $f(x) = \begin{cases} 4 - x^2 & \text{if } |x| < 2 \\ |x| - 2 & \text{if } |x| \ge 2 \end{cases}$ then f has minimum values when $x =$

a) 0 b) 2 c) 1 or -1 d) 2 or -2 e) -2

29) The equation of the tangent to the curve $y = x^3 - x^2 + 2x$ at its point of inflection is:

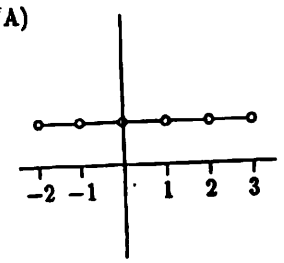
a) $y = 2x$ b) $23x - y = 45$ c) $45x - 27y = -1$ d) $9x - 27y = -13$ e) $63x - 27y = 5$

30) If $f(x) = 3x^5 - 10x^4 + 7x - 5$, then there are inflection points when $x =$

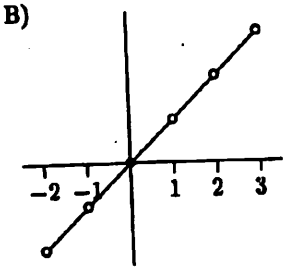
a) 0 b) 2 c) 0 or 2 d) -2 e) 0 or -2

31. Which graph of f best suits the description of f over $-2 \le x \le 3$.
$f'(n)$ is undefined for n any natural number.
$f'(x) = 1$ for any $x \ne n$.

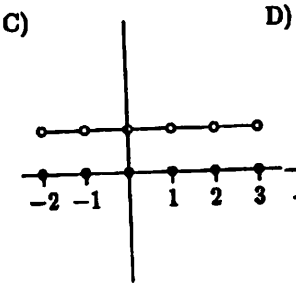
A)



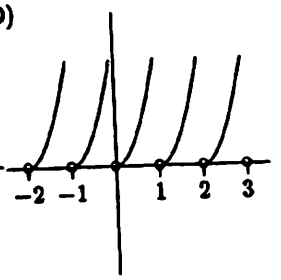
B)



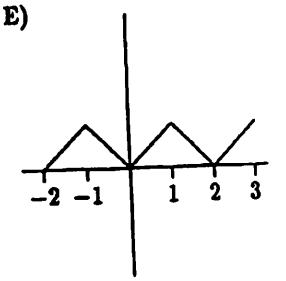
C)



D)

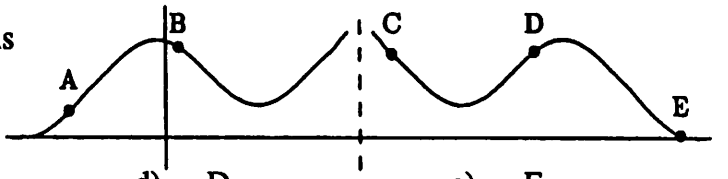


E)



32. At which of the five points on the graph on the right is

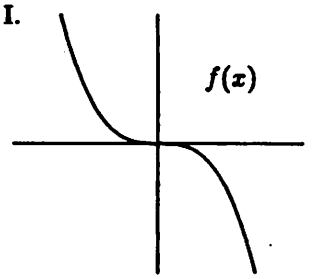
$$\frac{dy}{dx} > 0 \text{ and } \frac{d^2y}{dx^2} < 0$$



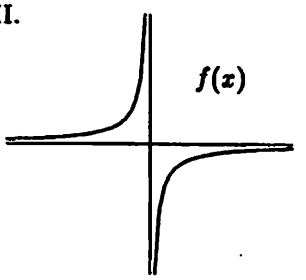
a) A b) B c) C d) D e) E

33. For which of the following functions is $f'(x) < 0$ whenever $f(x)$ defined?

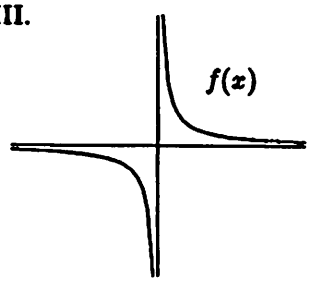
I.



II.

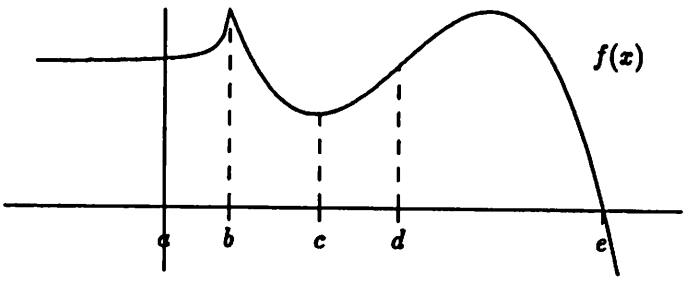


III.



a) I only b) II only c) III only d) I and II only e) I and III only

34. On the graph of f show on the right, at which indicated x-value does it appear that the rate of change in the slope of f is zero? At $x =$



a) A b) B c) C d) D e) E

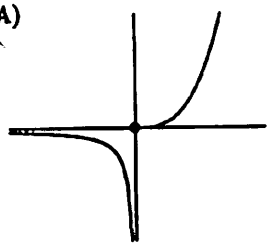
35. If $f(x) = x^4 + 4x^3 + 6x^2 + 4x$, the inflection point(s) is(are):

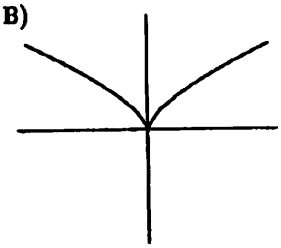
a) $(-1, 0)$ b) $(1, 16)$ c) $(0, 1)$ d) There are none. e) $(1, 16)$ and $(-1, 0)$

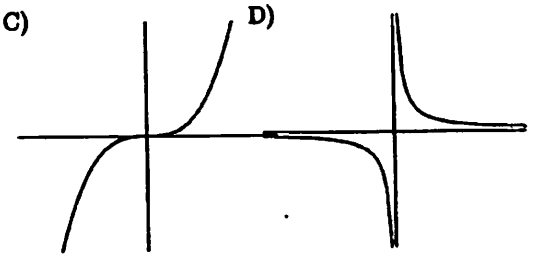
36. If $f(x) = \dfrac{x^2 - 1}{x^2}$, then the graph of f is concave up for:

a) All values of x b) $-1 < x < 1$ c) $x < -1$ or $x > 1$ d) No values of x e) $-\sqrt{2} < x < \sqrt{2}$

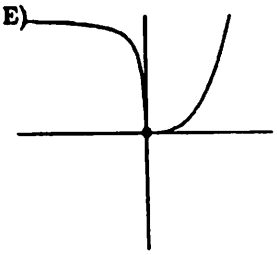
37. Which graph of f suits the description of f, a continuous function, where
$f(x)>0,\ f'(x)>0,\ f''(x)>0$ for $x>0$ **and** $f(x)>0,\ f'(x)<0,\ f''(x)<0$ for $x<0$.

A) 

B) 

C) 

D)

E) 

38. If f is a continuous function with $f''(x)=-5(3+2x)(2-3x)^2(3x-1)^3$, find the set of values of x for which f has an inflection point.

a) $\left\{-\frac{3}{2},\frac{1}{3},\frac{2}{3}\right\}$ b) $\left\{0,-\frac{3}{2},\frac{1}{3},\frac{2}{3}\right\}$ c) $\left\{-\frac{3}{2},\frac{1}{3}\right\}$ d) $\left\{-\frac{3}{2}\right\}$ e) $\left\{\frac{1}{3}\right\}$

39) Let $f(x)=2x^3+3x^2-36x$. The point $(2,-44)$ is a relative minimum because:

I. $f'(2)=0$ II. $f''(2)<0$ II. $f(2)<0$

a) I only b) II only c) III only d) I and II only e) I, II, and III

40. The relative minimum of $g(x)=\dfrac{3x^2+3}{x}$ is $x=1$ because:

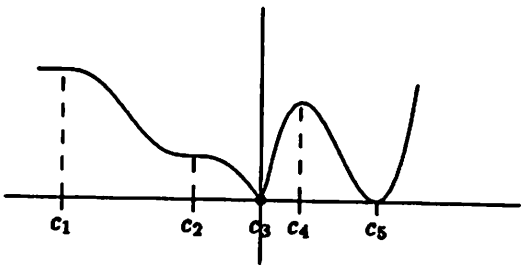
I. $g'(1)=0$ II. $g''(1)>0$ III. $g(1)=6$

a) I only b) II only c) III only d) I and II only e) I, II, and III

41. If $f(x)=\dfrac{x^2+1}{x}$, then f has a relative minimum when $x=$

a) 0 b) 1 c) -1 d) no minimum e) $\sqrt{3}$

42. On the graph, which indicated x-value would satisfy: $f''(x)>0$ when $f'(x)=0$?



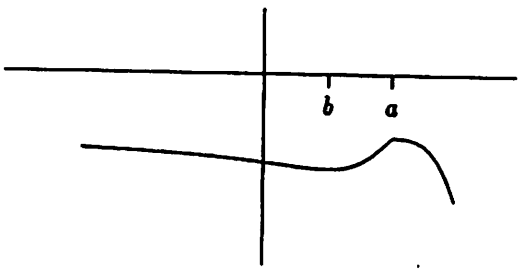
a) c_1 b) c_2 c) c_3 d) c_4 e) c_5

43. Given the graph of f, which of the following statements are true?

I. $f''(x)<0$ when $x>a$.

II. $f'(x)<0$ when $x>b$.

III. $f'(a)=0$.



a) I only b) II only c) III only d) I and II only e) I, II, and III

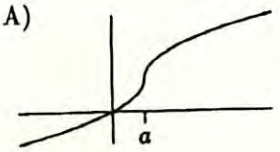
44. For the function $f(x)=\sqrt[3]{x}$, find $\lim_{x\to\infty} f'(x)$.

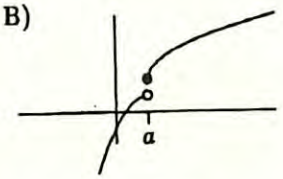
a) ∞ b) 0 c) 1 d) 3 e) 2

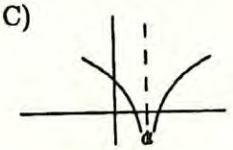
45. The function $f(x)=\dfrac{x^3-2x+1}{(1-x)^2}$ has an oblique asymptote:

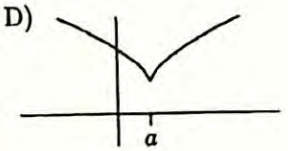
a) $y=x$ b) $y=x-2$ c) $y=x+2$ d) $y=2-x$ e) $y=x-3$

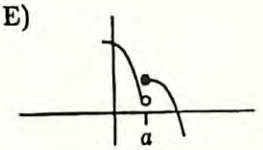
46. If f is continuous and differentiable for all x except $x = a$ and always concave downward, then which of the following could be a graph for f?

A) 

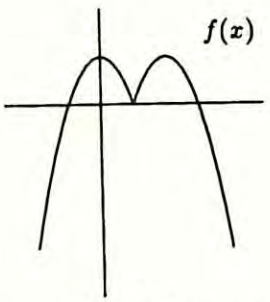
B) 

C) 

D) 

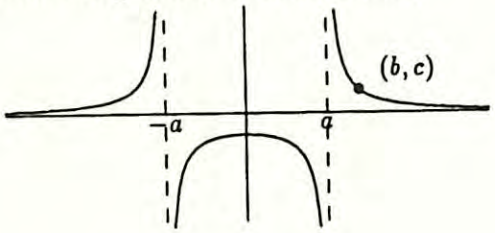
E) 

47. For the graph of f, which of the following statements is true?



a) $f(x)$ is continuous.
b) $f(x)$ is differentiable for all x.
c) $f(x)=|f(x)|$
d) $f(x)$ is an even function.
e) $f(x)$ has no relative minimum point.

48. For the function f with the given graph, which of the following statements is not true?

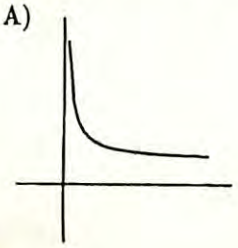


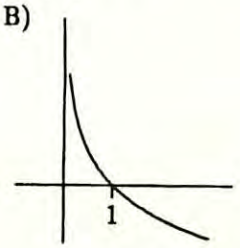
a) $f(x)$ is discontinuous at $-a$ and a.
b) $f'(x)$ is positive if $x<0$.
c) There is a relative maximum at $x = 0$.
d) $f''(x)$ is positive if $-a<x<a$.
e) $\lim_{x\to -b} f(x)=c$.

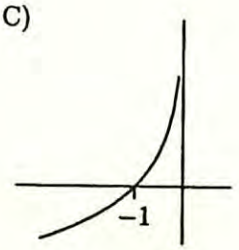
49. $\lim_{x\to\infty} \ln\left(1+\dfrac{1}{x}\right)^{2x} =$

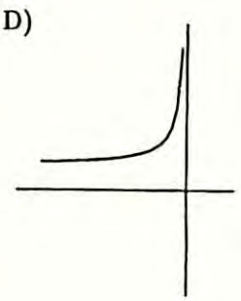
a) 2 b) ∞ c) 1 d) $2e$ e) e^2

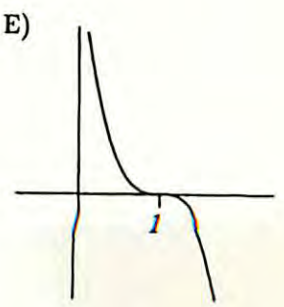
50., The graph of $y=\ln\left(\dfrac{1}{x}\right)$ is :

A) 

B) 

C) 

D) 

E) 

51. $\lim_{x \to \pi/4} \dfrac{\sin\left(x - \frac{\pi}{4}\right)}{x^2 - \pi^2/16} =$

a) 0 b) $\frac{2}{\pi}$ c) $\frac{1}{8\pi}$ d) $\frac{8}{3\pi}$ e) Nonexistent

52. $\lim_{x \to \infty} \dfrac{\ln\left(x^3\right)}{e^{2x}} =$

a) 0 b) ∞ c) 1 d) $\frac{1}{2}$ e) $\frac{3}{2}$

53. In the invalid argument:

$$\lim_{x \to 0} \frac{x^6 - x^5 + x^4 - x^3}{x^5 - x^4 + x^3 - x^2} = \lim_{x \to 0} \frac{6x^5 - 5x^4 + 4x^3 - 3x^2}{5x^4 - 4x^3 + 3x^2 - 2x} \quad \textbf{Step 1}$$

$$\lim_{x \to 0} \frac{30x^4 - 20x^3 + 12x^2 - 6x}{20x^3 - 12x^2 + 6x - 2} \quad \textbf{Step 2}$$

$$\lim_{x \to 0} \frac{120x^3 - 60x^2 + 24x - 6}{60x^2 - 24x + 6} \quad \textbf{Step 3}$$

$$\lim_{x \to 0} \frac{360x^2 - 120x + 24}{120x - 24} \quad \textbf{Step 4}$$

$$\lim_{x \to 0} \frac{720x - 120}{120} \quad \textbf{Step 5}$$

At what step should one stop using L'Hôpital's Rule and evaluate the limit? Step:

a) 1 b) 2 c) 3 d) 4 e) 5

54. Evaluate $\lim_{x \to \frac{\pi}{2}^-} \left(\dfrac{\sin^2 x}{1 - \cos x} \right)^{\tan x}$

a) 0 b) 1 c) $\frac{\pi}{2}$ d) ∞ e) e

55. Evaluate $\lim_{x \to 0} \dfrac{x - \tan x}{\sin x - x}$

a) 2 b) 0 c) ∞ d) 1 e) $\frac{1}{2}$

56. Evaluate $\lim_{x \to \infty} \dfrac{e^x}{x^2 + 4}$

a) There is no limit b) $\frac{1}{4}$ c) 0 d) 1 e) $\frac{1}{2}$

57. $\lim_{x \to 0} \dfrac{\sin^{-1} x}{\tan^{-1} 2x}$

a) ∞ b) $-\infty$ c) 0 d) 1 e) $\frac{1}{2}$

58. $\lim_{x \to \infty} x^{1/x} =$

a) ∞ b) 1 c) e d) $\frac{1}{e}$ e) 0

59. Evaluate $\lim_{x \uparrow \pi/2} \dfrac{\sec x}{\tan x}$.

a) ∞ b) $-\infty$ c) 1 d) -1 e) 0

60. Evaluate $\lim_{x \to \infty} [\ln x - \ln(1+x)]$.

a) There is no limit. b) $-\infty$ c) -2 d) 0 e) 1

61. $\lim_{x \to \infty} \left(\sqrt{x^2 + 2x} - x\right) =$

a) 0 b) $\frac{1}{2}$ c) 1 d) 2 e) The limit does not exist.

62. $D_x\left[\ln \sec^{-1} x^2\right] =$

a) $\dfrac{1}{\sec^{-1} x^2}$ b) $\dfrac{2x}{\sec^{-1} x^2}$

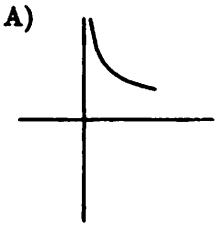
c) $\dfrac{1}{\left(\sec^{-1} x^2\right)\left(x^2\sqrt{x^4 - 1}\right)}$ d) $\dfrac{2}{x\sqrt{x^2 - 1}\left(\sec^{-1} x^2\right)}$ e) $\dfrac{2}{x\sqrt{x^4 - 1}\left(\sec^{-1} x^2\right)}$

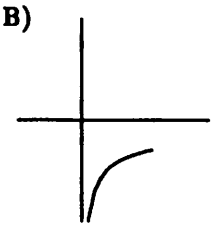
63. $\dfrac{d}{dx}\left[\tan 3^{2x}\right] =$

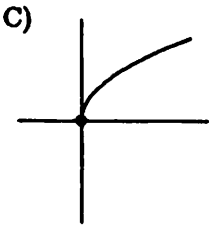
a) $\sec^2 3^{2x}$ b) $\sec^2 3^{2x} \ln 3$ c) $\left(\sec^2 3^{2x}\right)3^{2x}$

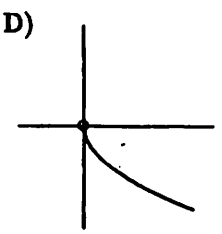
d) $2\left(3^{2x}\right)\left(\sec^2 3^{2x}\right)\ln 3$ e) $\left(3^{2x}\right)\left(\sec^2 3^{2x}\right)\ln 3$

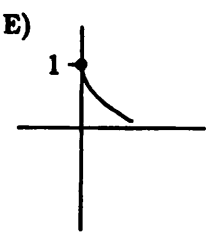
64. $\lim_{x \to 0^+} \dfrac{\ln x}{x}$ behaves graphically near zero like:

A) 

B) 

C) 

D) 

E) 

Practice Edition

$y = x^2$
$2x$
2

① $f(x) = \ln x \quad 1 \le x \le 4$

$$\frac{\ln 4 - \ln 1}{4-1} = \frac{\ln 4}{3} \overset{\text{set}}{=} f'(x) > \frac{1}{x}$$

$$x = \frac{3}{\ln 4} \quad (B)$$

② $y = \sqrt{x}^{\sqrt{x}}$

$$\ln y = \sqrt{x} \ln \sqrt{x}$$

$$\frac{1}{2}\sqrt{x}\ln x = \frac{\sqrt{x}\ln x}{2} \quad (B)$$

③ $y = \cos^2 x - \sin^2 x$

$= \cos 2x$

$y' = -2\sin 2x$

$y'' = -4\cos 2x$ (E)

④ $y = 3x^{100}$

$y^{100} = 3(100!)$ A

⑤ $f(x) = \sqrt{x^3}$

$= x^{\frac{3}{2}}$

$f' = \frac{3}{2}x^{\frac{1}{2}} \quad f'' = \frac{3}{4}x^{-\frac{1}{2}} \quad f''' = -\frac{3}{8}x^{-\frac{3}{2}}$

$x = \frac{1}{4}$

$-\frac{3}{8}\left(\frac{1}{\frac{1}{8}}\right) = -3$ A

⑥ $f(x) = (\sin x)^2$

$2\sin x \cos x = \sin 2x$

$f'' = 2\cos 2x$ (E).

(7) $f(x) = \frac{1}{x} = x^{-1}$

$f' = -x^{-2}$
$f'' = 2x^{-3}$
$f''' = -6x^{-4}$
$f^{IV} = 24x^{-5}$

$f^{V} = 5!\,x^{-6}$
$f^{VI} = 6!\,x^{-7}$ (B)

(8) $f(x) = \frac{3x-1}{x+2}$ on $[0,2]$

$f'(x) = \frac{3(x+2)-(3x-1)}{(x+2)^2}$

$= \frac{3x+6-3x+1}{(x+2)^2}$

$\frac{7}{(x+2)^2} > 0$

cont. & f(x) increasing $\therefore f(0) = -\frac{1}{2}$ is min value. (C)

(9) $f(x) = 3x^3 - \frac{9}{2}x^2$ $[-2,2]$

$f'(x) = 9x^2 - 9x$

$9x(x-1)$

sign of f': $+$ on $[-2,0]$, $-$ on $[0,1]$, $+$ on $[1,2]$

(0,0)

$3(8) - \frac{9}{2}\cdot 4$

$24 - 18$

max at $x=2$. increases to it 6 (B)

(10) $f(x) = 2x^3 + 3x^2 - 12x + 1$ $[-1,2]$

$f'(x) = 6x^2 + 6x - 12$

$6(x^2+x-2)$

$6(x+2)(x-1)$

f': $+$ on $[-2,-1]$, $-$ on $[-1,1]$, $+$ on $[1,2]$

max at $x=-1$ def'n it
max at $x=2$ INC to it

$f(2)$ $2\cdot 8 + 3\cdot 4 - 24 + 1$

$16 + 12 - 24 + 1$

5.

$f(-1) = -2 + 3 + 12 + 1$

14

(B) Abs Max 14 at $x=-1$

(11) $\lim_{x\to\infty} (3+2e^x)^{\frac{3}{x}}$

Let $y = (3+2e^x)^{\frac{3}{x}}$

$\ln y = \frac{3}{x}\ln(3+2e^x)$

$\lim_{x\to\infty} \ln y = \frac{3\ln(3+2e^x)}{x}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \ln y = \frac{3\cdot\frac{2e^x}{3+2e^x}}{1}$

$\lim_{x\to\infty} \ln y = \frac{6e^x}{3+2e^x}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \ln y = \frac{6e^x}{2e^x} = 3$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 3$

$\ln y = 3$

$\therefore e^3 = y$

(D)

(11) $\lim_{x\to\infty} (3+2e^x)^{3/x}$

Let $y = (3+2e^x)^{\frac{3}{x}}$

$\ln y = \frac{3}{x}\ln(3+2e^x)$

$\lim_{x\to\infty} \ln y = \frac{\lim (3+2e^x)}{\frac{x}{3}}$ $\frac{\infty}{\infty}$ l'hopital

$= \frac{\frac{2e^x}{3+2e^x}}{\frac{1}{3}} = \frac{6e^x}{3+2e^x}\,\frac{\frac{1}{e^x}}{\frac{1}{e^x}} = \frac{6}{\frac{3}{e^x}+2} = 3$

$\ln y = 3 \therefore e^3 = y$ D

$\left(1+\frac{2e^x}{3}\right)^{\frac{3}{2e^x}\cdot\frac{2e^x}{x}}\cdot 3^{\frac{3}{x}}$

$\lim 00^0 = e^3$ See Parents page.

(12) $f(x) = |x-4|^2$ $f''(c) = ?$

I: $x<4$ $[-(x-4)]^2$ $(x-4)^2$

II: $x\geq 4$ $(x-4)^2$ $(x-4)^2$

$f(x) = (x-4)^2$

$f'(x) = 2(x-4) \overset{\text{Set}}{=} 0$

$x = 4$

$f''(x) = 2$

$f''(\text{anything}) = 2$

$\therefore$ B

(13) $f(x) = (2x^2-8)^{\frac{4}{3}}$

$f'(x) = \frac{4}{3}(2x^2-8)^{\frac{1}{3}}\,4x$

$\frac{16}{3}x(2x^2-8)^{\frac{1}{3}} \geq \frac{16x}{3}(2(x-2)(x+2))^{\frac{1}{3}}$ (A)

Increasing $-2<x<0$ or $x>2$.

$-$ 0 $+$ 0 $-$ 0 $+$

-2 0 2

(14) $f(x) = x^3+ax^2$

$f'(x) = 3x^2+2ax$

$x(3x+2a) \overset{\text{Set}}{=} 0$

$3x+2a\big|_{x=1} = 3+2a \overset{\text{Set}}{=} 0$

$a = -\frac{3}{2}$ (A).

(15) $f(x) = x^3 - 9x^2 \quad 0 \le x \le 10$

$f'(x) = 3x^2 - 18x$

$f''(x) = 6x - 18 \overset{\text{set}}{=} 0$

$6(x-3) = 0$

$x = 3$

f''	−	0	+
		3	

min at $x=0$ dec fm left,
max at $x=10$ inc to it

$f'(0) = 0$

$f'(10) = 3x(x-6)$

$30(4) = 120$ max slope of f, at $x=10$. (E)

(16) $f(x) = x^{\frac{1}{3}} - x^{\frac{2}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{2}{3}x^{-\frac{1}{3}} \overset{\text{set}}{=} 0$

$x^{-\frac{2}{3}} = 2x^{-\frac{1}{3}}$

$x^{\frac{1}{3}} = 2x^{\frac{2}{3}}$

$x = 8x^2$

$8x^2 - x = 0$

$x(8x-1) = 0$

$x = 0$ reject $\quad x > \frac{1}{8}$ (B)

to get this to graph on TI 82
$Y_1 = x^{\frac{1}{3}}(1 - x^{\frac{1}{3}})$
window $x[-2, 2]$ $[-2.5, 1.5]$

	+	0	−	0	+
		0		$\frac{1}{8}$	

tangent line goes vertical at $x=0$.

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} - \frac{2}{3x^{\frac{1}{3}}}$

it f' DNE at $x=0$.

(17) C_3 (E)

Slope of chord fm $(a, f(a))$ to $(b, f(b))$ is negative only C_3 has neg slope.

(18) $f(x) = x^{\frac{2}{3}} - 1 \quad [-1, 1]$

$f(\pm 1) = 0$

$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} \overset{\text{set}}{=} 0$

f' NOT cont at $x=0$. (E)

(19) I $f(x) = x^3 + 1$ $[-1, 1]$ II $f(x) = 1 - |x|$ $[-1, 1]$

III $f(x) = x^2 - x$ $[0, 1]$

Rolle's Thm requires that f' exist throughout the interval

I NO. $f(-1) = 0$, $f(1) = 2$

II NO Der DNE at $x = 0$ III OK

III only

C

(20) $f(x) = |3x - 2|$ on $(0, 1]$

$f'(\frac{2}{3})$ DNE. D

(21) D $f(a) = f(b)$ and $f'(c) = 0$.

A, B Der DNE at $x = c$

C, $f(a) \neq f(b)$

$f'(c) \neq 0$

(22) $f(x) = x^3 - 2x^2$ $[0, 2]$

$3x^2 - 4x \overset{\text{set}}{=} \frac{0 - 0}{2}$

$x(3x - 4) = 0$

$x \in \{0, \frac{4}{3}\}$.

$c \in (0, 2)$

$\therefore c = \frac{4}{3}$ only

(B)

(23) $f(x) = -\frac{1}{2}x^4 + \frac{1}{3}x^3 + 3x^2 - 4$

$f'(x) = -2x^3 + x^2 + 6x$

$-x(2x^2 - x - 6)$

$-x(2x + 3)(x - 2) \overset{\text{set}}{=} 0$

f increases $x < -\frac{3}{2}$ or $0 < x < 2$

(D)

(24) $H'(x) = (x - 1)(x + 2)(x - 3)^2$

max at $x = -2$ (B)

f' test.

(25) $f(x) = 3bx^2 + ax - 5 \quad (2, -3)$

$f(2) = 12b + 2a - 5 = -3$

$f'(x) = 6bx + a \overset{\text{set}}{=} 0$

$x = -\frac{a}{6b} \overset{\text{set}}{=} 2$

$12b + 2a = 2$

$-12b - a = 0$

$a = 2$

$12b = -a$

$-12b - a = 0$

$12b = -2$

$b = -\frac{1}{6}$ (C)

(26)

$H'(x) = 3x(x-5)(2x-3)^2(3x-4)^4$

$+ \; 0 \; - \; \textcircled{} \; - \; \textcircled{} \; 0 \; +$

$0 \quad \frac{4}{3} \quad \frac{3}{2} \quad 5$

max at 0

min at 5

2 Extrema. (C)

(27) $f(x) = \begin{cases} \sqrt{25 - x^2} & -5 \le x \le 5 \\ x - 5 & \text{if } x > 5 \\ -x - 5 & \text{if } x < 5 \end{cases}$

$f'(x) = \begin{cases} \frac{-x}{\sqrt{25 - x^2}} & \leftarrow \text{when } -5 < x < 0 \; f' > 0 \\ 1 & \text{if } x > 5 \leftarrow f' > 0 \\ -1 & \end{cases}$

$\therefore$ f increasing $-5 < x < 0$, $x > 5$ (C)

(28) $f(x) = \begin{cases} 4 - x^2 & \text{if } |x| < 2 \\ |x| - 2 & \text{if } |x| \ge 2 \end{cases}$

$f(x) = \begin{cases} -x - 2 & \text{if } x \le -2 \\ 4 - x^2 & -2 < x < 2 \\ x - 2 & \text{if } x \ge 2 \end{cases}$

$f'(x) = \begin{cases} -1 \\ -2x \\ 1 \end{cases}$

$- \; + \; 0 \; - \; +$

$f' \quad -2 \quad 0 \quad 2$

min $x = \pm 2$

f test.

Dn DNE at $x = -2, 2$ $\therefore$ must use f test. (D)

(29) $y = x^3 - x^2 + 2x$ $y' = 3x^2 - 2x + 2$ $y'' = 6x - 2 \overset{\text{set}}{=} 0$

$x = \frac{1}{3}$

$y|_{x=\frac{1}{3}} = \frac{1}{27} - \frac{1}{9} + \frac{2}{3}$ $y'|_{x=\frac{1}{3}} = \frac{1}{3} - \frac{2}{3} + 2$

$= \frac{1 - 3 + 18}{27}$ $= \frac{1 - 2 + 6}{3}$

$= \frac{16}{27}$ $y' = \frac{5}{3} = m$

$y - y_1 = m(x - x_1)$ $27y - 16 = 45x - 15$

$y - \frac{16}{27} = \frac{5}{3}\left(x - \frac{1}{3}\right)$ $45x - 27y + 1 = 0$

$\left(y - \frac{16}{27} = \frac{5}{3}x - \frac{5}{9}\right)27$ (C)

(30) $f(x) = 3x^5 - 10x^4 + 7x - 5$

$f'(x) = 15x^4 - 40x^3 + 7$

$f''(x) = 60x^3 - 120x^2$

$60x^2(x - 2)$

f'' $-$ 0 $-$ 0 $+$

0 2

only inflection pt is at $x = 2$. concavity changes, (B)

(31) B $f' = 1$ Ellipse Slope $+1$ and -1

(32) $\frac{dy}{dx} > 0$ f increasing $\frac{d^2y}{dx^2} < 0$ concave down

(A)

(33) $f'(x) < 0 \ \forall x \in D_f$ decreasing I No $f' = 0$ or $x = 0$

II No III Yes.

III only. (C).

(34) rate of change of slope = 0 is when we have PI on f ∴ (D)

(35) $f(x) = x^4 + 4x^3 + 6x^2 + 4x + 1$

$f'(x) = 4x^3 + 12x^2 + 12x + 4$

$f''(x) = 12x^2 + 24x + 12$ No change in concavity

$12(x^2 + 2x + 1)$

$12(x+1)^2$ + ⊗ + ∴ D

-1 0

(36) $f(x) = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$

$f'(x) = \frac{2}{x^3}$

$f''(x) = -\frac{6}{x^4} < 0 \ \forall x$ f Not concave up (D)

(37) f Positive, increasing concave up when $x > 0$

A D E

f Positive, decreasing concave down when $x < 0$

Not A, Not D (E)

(38) $f''(x) = -5(3+2x)(2-3x)^2(3x-1)^3$

− 0 + 0 − ⊗ −

$-\frac{3}{2}$ 0 $\frac{1}{3}$ $\frac{2}{3}$ (C)

Changes of concavity $x = \frac{-3}{2}, \frac{1}{3}$

(39) $f(x) = 2x^3 + 3x^2 - 36x$ (2, -44) rel min

$f'(x) = 6x^2 + 6x - 36$ + 0 − 0 +

$6(x^2 + x - 6)$ -3 0 2

$6(x+3)(x-2)$ $f'(2) = 0$ Yes (A)

$f''(x) = 12x + 6 \Big|_{x=2} > 0$ $f''(2) < 0$ NO

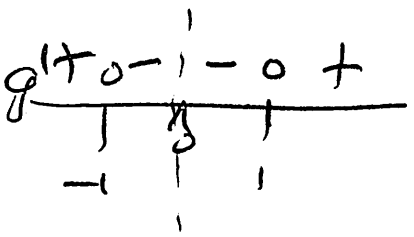
$f(2) < 0$ true but not necessary.

(40) $g(x) = \frac{3x^2+3}{x} = 3x + \frac{3}{x}$

$g'(x) = 3 - \frac{3}{x^2} \overset{\text{set}}{=} 0$

$x^2 = 1$

$x = \pm 1$



$g'(1) = 0$ yes,

$g''(1) > 0$ yes (44)

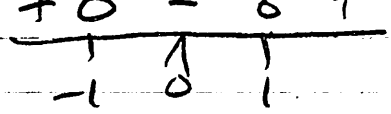
$g(1) = 6$ True but not necessary.

I, II only. D

(41) $f(x) = \frac{x^2+1}{x} = x + \frac{1}{x}$

$f'(x) = 1 - \frac{1}{x^2} \overset{\text{set}}{=} 0$

$x = \pm 1$



min at $x=1$ f'(0) (13)

(42) $f'' > 0$ concave up when f has max, min, P.I

c_5 (E) Note! at c_2 $f'' = 0$

(43) I concave down when $x > a$ Yes

II f decreasing when $x > b$ No

III $f'(a) = 0$ No Probably not defined. (A)

I only.

(44) $f(x) = \sqrt[3]{x}$ $\lim_{x \to \infty} f'(x)$

$f'(x) = \frac{1}{3} x^{-\frac{2}{3}}$ $\lim_{x \to \infty} f'(x) = 0$. (B).

(45) $f(x) = \frac{x^3 - 2x + 1}{(1-x)^2}$ $\frac{x^3 - 2x + 1}{x^2 - 2x + 1}$

$$\begin{array}{r} x+2 \\ x^2-2x+1 \;\overline{)\; x^3 - 2x + 1} \\ \underline{x^3 - 2x^2 + x} \\ 2x^2 - x \\ \underline{2x^2 + 4x - 2} \end{array}$$

$y = x + 2$. (C)

(46) (D)

(47) 4 only

B) Not at Pt.

C) No f is <0

D) No Not reflected in y axis

E) No min at pt.

(48)

A) Cont

B) $f' > 0$ $\forall x < 0$ True

C) rel max at $x=0$ True

(D) $f'' > 0$ NO. $f'' < 0$ $|x| < a$

E) True by def'n.

(49) $\lim_{x \to \infty} \ln\left(1+\frac{1}{x}\right)^{2x}$

$\lim_{x \to \infty} 2\ln\left(1+\frac{1}{x}\right)^{x} = 2\ln e = 2$ (A)

(50) $y = \ln\left(\frac{1}{x}\right) = \ln(x)^{-1} = -\ln x$

(B).

(51) $\lim_{x\to\frac{\pi}{4}} \frac{\sin\left(x-\frac{\pi}{4}\right)}{x^2-\frac{\pi^2}{16}}$ Let $t = x-\frac{\pi}{4}$, $x = t+\frac{\pi}{4}$ l'hospital $\frac{0}{0}$

$\lim_{x\to\frac{\pi}{4}} \frac{\sin t}{t\left(t+\frac{\pi}{4}+\frac{\pi}{4}\right)}$ $\left(x-\frac{\pi}{4}\right)\left(x+\frac{\pi}{4}\right)$ OR $\lim_{x\to\frac{\pi}{4}} \frac{\cos\left(x-\frac{\pi}{4}\right)}{2x}$

$\frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$ (B) $\frac{1}{2\cdot\frac{\pi}{4}} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$

(52) $\lim_{x\to\infty} \frac{\ln(x^3)}{e^{2x}}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\frac{3}{x}}{2e^{2x}} = \lim_{x\to\infty} \frac{3}{2xe^{2x}} = 0$ A

(53) $\frac{0}{0} = \lim_{x\to 0} \frac{6x^5-5x^4+4x^3-3x^2}{5x^4-4x^3+3x^2-2x}$ $\frac{0}{0}$ step 1

$=$ _____

$-2 \leftarrow$ no zeros on bottom

step 2 (B)

(54) $\lim_{x\uparrow\frac{\pi}{2}} \frac{x}{\cos x}$ Problem Replaced.

$\lim_{x\uparrow\frac{\pi}{2}} \frac{x\sin x}{\cos^2 x}$ $\frac{\frac{\pi}{2}}{0}$ — No limit $+\infty$ D

(55) $\lim_{x\to 0} \frac{x-\tan x}{\sin x - x}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{1-\sec^2 x}{\cos x - 1}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{-2\sec^2 x\tan x}{-\sin x}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{-2}{-\cos^2 x}\cdot\frac{\sin x}{\cos x\cdot\sin x}$

$\frac{2}{1} = 2.$

(4)

$$\frac{3}{(x+5)(x-1)} - \frac{2}{x+5}$$

$$\frac{3 - 2(x-1)}{(x+5)(x-1)}$$

$$\frac{-2x+5}{(x+5)(x-1)}$$

$$\frac{2(x+5)}{(x+5)(x-1)} = \frac{a}{(x+5)(x-1)}$$

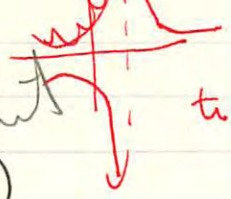
$$2x+10 = a$$

(56) $\lim_{x\to\infty} \frac{e^x}{x^2+4}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{e^x}{2x}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{e^x}{2} = +\infty$ No limit (A)

∞ is different than saying "No limit"



$\lim_{x\to 1} f(x)$ has no limit.

to say limit $\Rightarrow \infty$ is saying no limit but it also says where the limit is heading

(57) $\lim_{x\to -5} \frac{3}{x^2+4x-5} - \frac{2}{x-5}$

$\frac{3(x-5) - 2(x^2+4x-5)}{(x-5)(x+5)(x-1)}$

$(x+5)(x-1)$

$3x-15-2x^2-8x+10$

$-2x^2-5x-5$

$\frac{3x-15-2x^2-8x+10}{(x-5)(x+5)(x-1)}$

$-\frac{(2x^2+5x+5)}{(x-5)(x+5)(x-1)} = \frac{(2x^2+5x+5)}{(x-5)(x+5)(x-1)}$

Doesn't reduce. No Limit $\frac{+1}{(-10)(-5)}$ $-\infty$ (B)

Replaced

(2) $\lim_{x\to 0} \frac{\sin^{-1} x}{\tan^{-1} 2x}$ $\frac{0}{0}$

$\lim_{x\to 0} \frac{\frac{1}{\sqrt{1-x^2}}}{\frac{2}{1+4x^2}}$ $\frac{1}{2}$

$\lim_{x\to 0} \frac{1+4x^2}{2\sqrt{1-x^2}} = \frac{1}{2}$ (D)

(58) $\lim_{x\to\infty} x^{\frac{1}{x}}$

$y = x^{\frac{1}{x}}$

$\ln y = \frac{1}{x}\ln x$

$\lim_{x\to\infty} \frac{\ln x}{x}$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$

$\ln y = 0 \therefore y = e^0 = 1$ (B)

(59) $\lim_{x\uparrow \frac{\pi}{2}} \frac{\sec x}{\tan x}$ $\frac{\infty}{\infty}$

$\frac{\frac{1}{\cos x}}{\frac{\sin x}{\cos x}}$

$\lim_{x\uparrow \frac{\pi}{2}} \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} =$

$\lim_{x\uparrow \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x}$

$\lim_{x\uparrow \frac{\pi}{2}} \frac{\tan x}{\sec x}$

$\lim_{x\uparrow \frac{\pi}{2}} \frac{1}{\sin x} = 1$

(C)

(60) $\lim_{x\to\infty}\left[\ln x - \ln(1+x)\right]$

$\lim_{x\to\infty}\left[\ln\left(\frac{x}{1+x}\right)\right]$ $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \ln\left(\frac{1}{1}\right) = 0.$ D

(61) $\lim_{x\to\infty} \sqrt{x^2+2x} - x \quad \frac{\sqrt{\quad}+x}{\sqrt{\quad}+x}$

$\lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$

$\lim_{x\to\infty} \frac{2x}{x\left(\sqrt{1+\frac{2}{x}}+1\right)}$

$\lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{2}{x}}+1} \to \frac{2}{1+1} = 1$ (C)

(62) $D_x\left[\ln \operatorname{Sec}^{-1} x^2\right]$

$\frac{1}{\operatorname{Sec}^{-1}x^2} \cdot \frac{1}{x^2\sqrt{x^4-1}} \cdot 2x$

$\frac{2}{x \operatorname{Sec}^{-1}x^2 \sqrt{x^4-1}}$ (E)

(63) $\frac{d}{dx}\left[\tan 3^{2x}\right]$ $\sec^2 3^{2x} \cdot d(3^{2x})$

$\cdot e^{2x\ln 3}$

$\cdot 3^{2x} \cdot 2\ln 3$

$3^{2x} \cdot 2\ln 3 \left(\sec 3^{2x}\right)$ D

(64) $\lim_{x\to 0^+} \frac{\ln x}{x}$ $\frac{-\infty}{0}$ Not L'hopital Situation.

Neg Large divided by Small = Very Neg.

(B)